Independent-Set $\leq$ Clique

**Definition**
Suppose $G = (V, E)$ is an undirected graph and $S \subseteq V$.

(a) $S$ is **independent** when for each $u, v \in S$, $(u, v) \notin E$.
(b) $U$ is a **clique** when for each distinct $u, v \in U$, $(u, v) \in E$.
(c) $\overline{G} = (V, E)$ where $E = \{(u, v) : (u, v) \notin E\}$.

**Lemma**

$S$ is an independent set in $G$ \iff $(V - S)$ is a clique in $\overline{G}$.

**Independent Set Problem**

Given: $G$ and $b$.

Find: An indep. set for $G$ of size $\geq b$.

**Clique Problem**

Given: $G$ and $b$.

Find: A clique in $G$ of size $\geq b$.

$f((V, E)) = ??$

$h(S) = ??$

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3D Matching $\leq$ ZOE

**3D Matching**

Given: $R \subseteq U \times V \times W$ where $|U| = |V| = |W| = n$.

Find: A size-$n$ subset $M \subseteq R$ such that

- $|\{u, v, w\} : (u, v, w) \in M\}$.
- $|\{u, v, w\} : (u, v, w) \notin M\}$.

**Zero-One Equations (ZOE)**

Given: $A$, an $m \times n$ matrix of 0’s and 1’s

Find: $\vec{x}$, an $n$-vector of 0’s and 1’s such that $A\vec{x} = \vec{1}$.

**Construction**

- Suppose $m = |R|$.
- $x_j$ variable for the $j$-th triple.
- $x_j = 1 \iff j$-th triple is used.
- $(A_{ij})$ is $3m \times n$
- $A_{ij} = 1 \iff j$-th eqn.
- $A_{ij} = 0 \iff j$-th eqn.
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So $A\vec{x} = \vec{1}$ means …
ZOE $\leq$ Subset Sum

Zero-One Equations (ZOE)

Given: A, an $m \times n$ matrix of 0’s and 1’s
Find: $\vec{x}$, a n-vec. of 0’s and 1’s $\exists A\vec{x} = \vec{1}$.

A Sample Reduction

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ll \begin{cases} M = \{ 1000_5, 0001_5, 0110_5, 1000_5, 0100_5 \} \\ G = 1111_5 \end{cases}$$

Subset Sum

Given: A multiset $M$ and goal $G$, all ints.
Find: An $M' \subseteq M$ such that $G = \sum_{x \in M'} x$.

ZOE $\leq$ ILP

Zero-One Equations (ZOE)

Given: $A$, an $m \times n$ matrix of 0’s and 1’s
Find: $\vec{x}$, a n-vec. of 0’s and 1’s $\exists A\vec{x} = \vec{1}$.

Int. Linear Programming (ILP)

Given: constraints $A\vec{x} \leq \vec{b}$
Find: A vector of integers $\vec{x} \exists A\vec{x} \leq \vec{b}$.

Reduction:
- Each ZOE equation: $\vec{a} \cdot \vec{x} = 1$
  is rewritten to two inequalities: $\vec{a} \cdot \vec{x} \leq 1$ and $-\vec{a} \cdot \vec{x} \leq -1$
- Each ZOE variable $x_i$ gets two additional constraints: $x_i \leq 1$ and $-x_i \leq 0$.

Rudrata/Hamiltonian Cycle $\leq$ TSP

Rudrata/Hamiltonian Cycle Problem

Given: $G = (V, E)$, an undirected graph.
Find: A simple cycle that visits each vertex of $G$.

Traveling Salesman Problem (TSP)

Given: $V'$, $n$ vertices; all $\frac{n(n-1)}{2}$-many distances between them; and $b$, a budget.
Find: a, an ordering of $1, \ldots, n$, such that $\sum_{i=1}^{n} d(x_{a(i)}, x_{a(1+i \mod n)}) \leq b$.

Construction Pick $a \geq 1$.
Given $(V, E)$, define

- $V' = V$
- $d_{ij} = \begin{cases} 1, & \text{if } (i, j) \in E; \\ 1 + a, & \text{otherwise.} \end{cases}$
- $b = |V|$

Claim: $(V, E)$ has a R/H cycle

If $a = 1$:
- Distances satisfy the triangle inequality: $d_{ij} + d_{jk} \geq d_{ik}$.
- These instance of TSP are approximatable (Chap. 9).

If $a \gg 1$:
- Gap: either a solution of cost $n$, or solutions with costs $\geq n + a$, but none inbetween.

All of NP $\leq$ Circuit SAT, 1

Circuit SAT

Given: A Boolean circuit, a dag with five types of gates
1. AND gates and Or gates (indegree 2)
2. NOT gates (indegree 1)
3. Known input gates (indegree 0) labeled True or False.
4. Unknown inputs gates (indegree 0) labeled ?.
5. An output gate (a particular sink)

Find: A truth assignment to the unknown input gates so that the circuit evaluates to True.

Claim 1: SAT $\leq$ Circuit SAT
All of NP $\leq$ Circuit SAT, 4

**Circuit SAT**

**Given:** A Boolean circuit, a dag with five types of gates
1. AND gates and Or gates (indegree 2)
2. NOT gates (indegree 1)
3. Known input gates (indegree 0) labeled True or False.
4. Unknown inputs gates (indegree 0) labeled “?”.
5. An output gate (a particular sink)

**Find:** A truth assignment to the unknown input gates so that the circuit evaluates to True.

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All of NP $\leq$ Circuit SAT, 5

**Theorem (The Cook-Levin Theorem)**

SAT is NP-complete.

Next: Dealing with NP-completeness

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Given an NP-problem Q with polytime checking function C.

- For instance I, build the boolean circuit B that checks I, with “?” labeling the “solution input gates.”
- Then I has a Q-solution $\iff$ B has a satisfying assignment.

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