Give a polynomial-time algorithm for $P$. Be sure to explain why your algorithm runs in polynomial-time.

**An answer to 2.** Here is the algorithm.

```plaintext
input I, an instance of P
result ← A_Q(f(I))
if result = “no answer” then return “no answer”
else return h(result)
```

Since $f$, $h$, and $A_Q$ are poly-time computable (and since polytime functions are closed under composition) the above is also poly-time.

**Problem 3. (10 points)**

This question concerns the following two search problems. (Definition 1 on page 2 may be helpful here.)

3D-Matching.

- **Given:** Sets $A$, $B$, and $C$ with $|A| = |B| = |C| = n$ and $T \subseteq A \times B \times C$.
- **Find:** $M \subseteq T$ with $|M| = n$ such that if $(a, b, c)$ and $(a', b', c')$ are distinct elements of $M$, then $a \neq a'$, $b \neq b'$, and $c \neq c'$.
- **Exact-Set-Cover.**
  - **Given:** $S$, a finite collection of finite sets. and $K$, a positive integer.
  - **Find:** A pairwise disjoint subcollection $\{S_1, \ldots, S_K\} \subseteq S$ that covers $\cup S$.

**a. (1 point)** Explain why Exact-Set-Cover is in NP.

**An answer to 3a.** Here is a checking algorithm for Exact-Set-Cover.

```plaintext
input (S, K) and \{ S_1, \ldots, S_m \}
Check that $m = K$ and $S_i \cap S_j = \emptyset$ for $i \neq j$.
Check that $\bigcup S = \bigcup_{i=1}^{m} S_i$
report whether all the checks were passed.
```

It is straightforward to show that the above is poly-time.

For parts b, c, and d, suppose $(A, B, C, T)$ is an instance of 3D-Matching, where $A$, $B$, and $C$ are pairwise disjoint. Let $S = \{ \{a, b, c\} : (a, b, c) \in T \}$ and $K = |A|$.
b. (1 point) Explain why $(S, K)$ is an instance of Exact-Set-Cover.

An answer to 3b. $S$ is a collection of sets and $K$ is a positive integer.

c. (2 points) Suppose $M$ is a solution to the 3D-Matching problem $(A, B, C, T)$. Let $S' = \{ a, b, c \} : (a, b, c) \in M \}$. Explain why $S'$ is a solution to the Exact-Set-Cover problem $(S, K)$.

An answer to 3c. By construction, $|S'| = n = K$; and if $\{ a, b, c \}$ and $\{ a', b', c' \}$ are distinct elements of $S'$ (and $a, a' \in A, b, b' \in B$, and $c, c' \in C$), then $(a, b, c), (a', b', c') \in M$, and by the conditions on 3D-Matching, $a \neq a', b \neq b'$, and $c \neq c'$ so $\{ a, b, c \} \cap \{ a', b', c' \} = \emptyset$ since $A, B, C$ are pairwise disjoint. Also since $\bigcup S = A \cup B \cup C = \bigcup S'$, $S'$ is a cover. Hence, $S'$ is a solution to $(S, K)$.

d. (2 points) Suppose $\{ S_1, \ldots, S_K \}$ is a solution to the Exact-Set-Cover problem $(S, K)$. Let $M = \{ (a, b, c) : \{ a, b, c \} = S_i, a \in A, b \in B, c \in C, 1 \leq i \leq K \}$. Explain why $M$ is a solution to the 3D-Matching problem $(A, B, C, T)$.

An answer to 3d. Since $K = |A| = |B| = |C|$, $M$ has the right number to triples. Since the $S_i$'s are pairwise disjoint, it follows that if $(a, b, c)$ and $(a', b', c')$ are distinct elements of $M$, then $a \neq a', b \neq b'$, and $c \neq c'$. Hence, $M$ is a solution to $(A, B, C, T)$.

e. (3 points) Show that 3D-Matching $\preceq$ Exact-Set-Cover. Be sure to:

(i) (1 point) say what the functions $f$ and $h$ are,
(ii) (1 point) explain why if $I$ is an instance of 3D-Matching and $S$ is a solution to $f(I)$, then $h(S)$ is a solution to $I$, and
(iii) (1 point) explain why if $I$ is an instance of 3D-Matching that has a solution, then $f(I)$ is an instance of Exact-Set-Cover that also has a solution.

You may assume that the sets $A$, $B$, and $C$ in each instance of 3D-Matching are pairwise disjoint.

An answer to 3e.
(i) $f(A, B, C, T) = (S = \{ a, b, c \} : (a, b, c) \in T, K = |A|)$.
$h(S') = \{ (a, b, c) : (a, b, c) \in S', a \in A, b \in B, c \in C \}$.
(ii) See part d. (Yes, that is all you needed to say.)
(iii) See part c. (Yes, that is all you needed to say.)

f. (1 point) Suppose we know that 3D-Matching is NP-complete. Using the above (and maybe some Math Facts) explain why Exact-Set-Cover is also NP-complete.

An answer to 3f. As 3D-Matching is NP-complete, every problem in NP reduces to it. By part e, 3D-Matching $\preceq$ Exact-Set-Cover. By Fact 1, $\preceq$ is transitive. Hence, every problem in NP reduces to Exact-Set-Cover. Therefore, by Definition 3a, Exact-Set-Cover is NP-hard. By part a, Exact-Set-Cover is in NP. Therefore, as Exact-Set-Cover is NP-hard and in NP, Exact-Set-Cover is NP-complete by Definition 3b.

Some Reference Math Facts

Definition 1. Suppose $\{ S_1, \ldots, S_m \}$ is a finite collection of finite sets.
a. $\bigcup \{ S_1, \ldots, S_m \} = \bigcup \bigcup S_m$.
b. $\{ S_1, \ldots, S_m \}$ covers a set $U$ when $\bigcup \{ S_1, \ldots, S_m \} \supseteq U$.

c. $\{ S_1, \ldots, S_m \}$ is a pairwise disjoint collection when $S_i \cap S_j = \emptyset$ for $i \neq j$.

Definition 2. For search problems $P$ and $Q$, $P \preceq Q$ means that there are polynomial-time computable functions $f$ and $h$ such that

- If $I$ is an instance of $P$, then $f(I)$ is an instance of $Q$ and if $S$ is a solution of $f(I)$, then $h(S)$ is a solution of $I$.
- If $I$ is an instance of $P$ that has a solution, then $f(I)$ is an instance of $Q$ that has a solution.

Definition 3.
a. $A$ is NP-hard means that $B \preceq A$ for every $B$ in NP.
b. $A$ is NP-complete means that $A$ is both:
(i) in NP and (ii) NP-hard.

Facts
1. $\preceq$ is transitive. That is, $|A \preceq B$ and $B \preceq C$ implies that $A \preceq C$.
2. If $A$ is NP-hard and $B \preceq A$, then $B$ is NP-hard.