ELE756 — Random Processes

Midterm Exam

Monday March 19, 2001

NAME

This is an open book open notes open everything exam. No open discussion though. For all problems, try to simplify your results or you'll lose a lot of points. Enjoy!

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1. * (25 points) Given a real random process $X(t)$ with joint (cumulative) distribution function $F_{X(t_1),\cdots,X(t_n)}(x_1,\cdots,x_n) = P[X(t_1) \leq x_1, \cdots, X(t_n) \leq x_n]$. Assume for a real number $a$ we define another binary random process

$$Y(t) = \begin{cases} 1 & X(t) \leq a \\ 0 & Y(t) > a \end{cases}$$

Prove that $E[Y(t)] = F_{X(t)}(a)$ and $R_Y(t_1, t_2) = F_{X(t_1), X(t_2)}(a, a)$, i.e., the mean and autocorrelation function of $Y(t)$ equal the one and two-dimensional distribution of the original random process $X(t)$ evaluated at $a$. 
2. ** (25 points) Assume we have a zero mean unit variance \( i.i.d. \) random sequence \( X(1), X(2), \ldots \).

   We define a new random sequence \( Y(n) \) as a cumulative sum of \( X(n) \), i.e.,
   \[
   Y(0) = 0 \\
   Y(1) = X(1) \\
   Y(2) = X(1) + X(2) \\
   \vdots \\
   Y(n) = X(1) + \cdots + X(n) = \sum_{i=1}^{n} X(i)
   \]

   (a) (15 points) Find \( \mu_Y(n) \) and \( R_Y(n_1, n_2) \) where \( n, n_1, \) and \( n_2 \) are all nonnegative integers. Is \( Y(n) \) a WSS random sequence?

   (b) (10 points) Now define another random sequence as
   \[
   Z(n) = \frac{Y(n + m) - Y(n)}{m}
   \]

   Where \( m > 0 \) is a fixed integer. Find \( \mu_Z(n) \) and \( R_Z(n_1, n_2) \). Is \( Z(n) \) WSS?
3. A white noise $X(t)$ with PSD $\mathcal{N}_0/2$ is passed through the following linear system where

\[
Y(t) = X(t) - X(t - T) \\
Z(t) = \int_{-\infty}^{t} y(u) du
\]

Find $R_Z(\tau)$ and $S_Z(\tau)$. 
4. *** Consider the following random process

\[ X(t) = A \cos(\omega_c t + \phi) + n(t) \]

where \( A, \omega_c \) and \( \phi \) are all constant, while \( n(t) \) is a bandpass white noise with PSD \( N_0/2 \), i.e.,

\[ S_N(\omega) = \begin{cases} \frac{N_0}{2} & |\omega \pm \omega_c| \leq W = 2\pi B \\ 0 & \text{Otherwise} \end{cases} \]

Now assume that \( X(t) \) is passed through the following system.

\[ \cos(\omega_c t) \]

\[ X(t) \]

\[ \times \]

\[ \text{LPF} \]

\[ Y(t) \]

where LPF is an ideal lowpass filter with bandwidth \( W \), i.e., its transfer function

\[ H(\omega) = \begin{cases} 1 & |\omega| \leq W = 2\pi B \\ 0 & \text{Otherwise} \end{cases} \]

(a) (15 points) Find the power spectral density of \( Y(t) \).

(b) (10 points) Now assume instead \( \phi \sim \text{uniform}(0, 2\pi) \), but \( A \) and \( \omega_c \) are still constant.

Calculate the average SNR (signal to noise power ratio) of the output signal \( Y(t) \). (Hint: Express SNR in terms of \( A, \phi, \) and \( N_0 \) and average over \( \phi \).)

**Useful trigonometric identity:** \( \cos \alpha \cos \beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)] \)