The standard first example of recursion: Factorial

\[\begin{align*}
0! &= 1 \\
1! &= 1 \\
2! &= 1 \times 2 \\
3! &= 1 \times 2 \times 3 \\
4! &= 1 \times 2 \times 3 \times 4 \\
5! &= 1 \times 2 \times 3 \times 4 \times 5 \\
\ldots \\
k! &= k \times (k-1)! & \text{where } k > 0
\end{align*}\]

```
-- fact n = n factorial
-- NOTE: n < 0 causes an error
fact :: Integer -> Integer
fact n
| n==0 = 1
| n>0 = n * fact (n-1)
```

```
fact 4
= \{ n = 4 \}
 4 * fact (4-1)
= \{ n = 3 \}
 4 * 3 * fact (3-1)
= \{ n = 2 \}
 4 * 3 * 2 * fact (2-1)
= \{ n = 1 \}
 4 * 3 * 2 * 1 * fact (1-0)
= \{ n = 0 \}
 4 * 3 * 2 * 1 * 1
```

The standard 2nd example of recursion: Fibonacci

\[\begin{align*}
f_0 &= 0 \\
f_1 &= 1 \\
f_n &= f_{n-1} + f_{n-2}, & \text{for } n > 1.
\end{align*}\]

```
-- Fibonacci numbers
fib :: Integer -> Integer
fib n
| n==0 = 0
| n==1 = 1
| n>1 = fib(n-1) + fib(n-2)
| otherwise = error "fib given negative argument"
```

```
fib 4
fib 3
fib 2
fib 1
fib 0
```

The Haudenosaunee (Iroquois) have similar rules for clan membership.

**Question:** Who counts as being Jewish?

**One answer:** Either:

a. You are Abraham, or
b. you are a convert, or
c. your mother was Jewish. (the recursive case)

```
Recursion on lists, 1

**Typical recursions on lists have at least two cases:**

1. The list you are recurring on looks like: \([\,]\)
   in which case, the recursion bottoms out (i.e., stops).

2. The list you are recurring on looks like: \((x:xs)\)
   in which case you probably have subcase where the recursion continues on \(xs\).

\[
\text{sum'} :: (\text{Num } a) \Rightarrow [a] \rightarrow a
\]
\[
\text{sum'} [] = 0
\]
\[
\text{sum'} (x:xs) = x + \text{sum'} xs
\]

Recursion on lists, 2

**A messier example**

\[
\text{maximum'} :: (\text{Ord } a) \Rightarrow [a] \rightarrow a
\]
\[
\text{maximum'} [] = \text{error "maximum of empty list!"} \quad (1)
\]
\[
\text{maximum'} [x] = x \quad (2)
\]
\[
\text{maximum'} (x:xs) = \text{max } x (\text{maximum'} xs) \quad (3)
\]

\[
\text{maximum'} [2,5,1]
\]
\[
= \quad \{(3) \text{ succeeds with } x = 2, xs = [5,1]\}
\]
\[
= \quad \{(3) \text{ succeeds with } x = 5, xs = [1]\}
\]
\[
= \quad \text{max } 2 (\text{max } 5 (\text{maximum'} [1]))
\]
\[
= \quad \{(2) \text{ succeeds with } x = 1\}
\]
\[
= \quad \text{max } 2 (\text{max } 5 1)
\]
\[
= \quad \text{max } 2 5
\]
\[
= \quad 5
\]

Recursions on lists, 3

**Class exercises**

- \(\text{replicate'} :: \text{Int} \rightarrow a \rightarrow [a]\)
  \(\text{replicate'} n x = \text{a list of } n \text{ copies of } x\)

- \(\text{take'} :: \text{Int} \rightarrow [a] \rightarrow [a]\)
  \(\text{take'} n xs = \text{the first } n \text{ elements of } xs\)

- \(\text{reverse'} :: [a] \rightarrow [a]\)
  \(\text{reverse'} xs = \text{the reverse of } xs\)

- \(\text{zip'} :: [a] \rightarrow [b] \rightarrow [(a,b)]\)
  \(\text{zip'} xs ys = \text{the zip of } xs \text{ and } ys\)

- \(\text{elem'} :: (\text{Eq } a) \Rightarrow a \rightarrow [a] \rightarrow \text{Bool}\)
  \(\text{elem'} x xs \text{ tests if } x \text{ is an element of } xs\)