Expressiveness of Fluted Logic

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Abstract

Fluted Logic is first-order predicate logic without variables. The lack of variables results in reduced expressiveness. Nevertheless, many logical problems that can be stated in natural language, such as the famous Schubert's Steamroller, can be rendered in fluted logic. Further evidence of the expressiveness of fluted logic is its close relation to description logics. It has been shown previously that fluted logic is decidable and has the finite-model property. This paper addresses the expressiveness of fluted logic. The paper concludes that fluted logic is not suitable for defining and reasoning about most mathematical entities. But it is an excellent formalism for construing natural language reasoning.

1 Introduction

Fluted formulas are first-order formulas without variables. Without variables, fluted formulas cannot permute the arguments of a predicate, or make distinct arguments equal, or add vacuous arguments.

One can also view a fluted formula as a first-order formula in which the order of the arguments of each predicate is precisely the order of the enclosing quantifier scopes. Thus in fluted formulas variables are redundant and simply could be eliminated.

In spite of these limitations, fluted formulas retain a significant part of the expressiveness of first-order formulas. Many logical problems that can be stated in natural language, such as the famous Schubert's Steamroller (Stickel [12]), can be construed in fluted formulas. Fluted logic (FL), the logic of fluted formulas, also has a close

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relation to description logics such as $\mathcal{ALC}$, to modal logic, and to path logic (e.g., see Ohlbach and Schmidt [5], Hustadt and Schmidt [4], Schmidt [10], Purdy [6]).

The syntax of fluted logic is extremely simple. Taking advantage of the fact that variables are dispensable in FL, the set of formulas $fml(L)$ of FL for a set of predicates L can be defined:

1. if $R \in L$, then $R \in fml(L)$;
2. if $\phi, \psi \in fml(L)$, then $\neg\phi, \phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi \in fml(L)$;
3. if $\phi \in fml(L)$, then $\exists\phi, \forall\phi \in fml(L)$.

For example, a sequence of numbers (N) are related under the successor relation (S) according to the formula: $\forall(N \rightarrow \exists(N \wedge S))$.

The objective of this paper is a characterization of the expressiveness of fluted formulas. It will be shown that each first-order structure determines a certain transition system. Given two structures, examination of their transition systems is sufficient to decide whether any fluted formula can distinguish between them. Based on this result, it will be shown that fluted formulas cannot express reflexivity, symmetry, or transitivity of relations. Thus while fluted formulas can specify a path in terms of properties of its points they cannot require that the path be acyclic. Hence fluted logic cannot differentiate between finite and infinite models. It follows that fluted logic is not well-suited for specifying most mathematical entities. However, it is well-suited for construal of natural language descriptions and reasoning about them.

2 Preliminaries

The definition of syntax and semantics of and inference in first-order predicate logic (FO) can be found in Andrews [1]. In this paper, the connectives are $\neg, \wedge, \vee$, and $\rightarrow$. The quantifiers are $\exists$ and $\forall$. The set of predicate symbols are those that occur in some given finite set of formulas called *premises*. The finite set of predicate symbols are referred to as the *lexicon*. If L is a lexicon and $R \in L$, then $ar(R)$ denotes the arity of R. $ar(L) := max\{ar(R) : R \in L\}$.

A subformula is *prime* if it is atomic or of the form $\exists x\zeta$ or $\forall x\zeta$.

A first-order *L-structure* $\mathcal{A}$ consists of a set A, the *domain*, and a mapping that assigns to each $R \in L$ a subset $R^{\mathcal{A}} \subseteq A^{ar(R)}$. The notions of satisfaction and truth are standard. If ψ is a formula over L with free variables among $\{x_1, \ldots, x_k\}$, and ψ is satisfied in $\mathcal{A}$ by the assignment of values to variables $\{x_i \mapsto a_i\}_{1\leq i\leq k}$, we write $\mathcal{A}, a_1 \cdots a_k \models \psi$. If ψ is a sentence and ψ is true in $\mathcal{A}$, we write $\mathcal{A}, \varepsilon \models \psi$ or simply $\mathcal{A} \models \psi$.

Let θ be a subformula of formula ϕ. The *polarity (positive or negative)* of θ is defined as follows.

1. If $\phi = \psi$ and θ is positive (negative) in ψ, then θ is positive (negative) in ϕ.

2. If $\phi = \neg\psi$ and θ is positive (negative) in ψ, then θ is negative (positive) in ϕ.

3. If $\phi = \psi \wedge \rho$ or $\phi = \psi \vee \rho$, then if θ is a subformula of ψ and θ is positive (negative) in ψ, then θ is positive (negative) in ϕ; if θ is a subformula of ρ and θ is positive (negative) in ρ, then θ is positive (negative) in ϕ.

4. If $\phi = \psi \rightarrow \rho$, then if θ is a subformula of ψ, and θ is positive (negative) in ψ, then θ is negative (positive) in ϕ; if θ is a subformula of ρ and θ is positive (negative) in ρ, then θ is positive (negative) in ϕ.

5. If $\phi = \exists x\psi$ or $\phi = \forall x\psi$ and θ is positive (negative) in ψ, then θ is positive (negative) in ϕ.

An important principle in first-order logic is the *Principle of Monotonicity*, embodied in the following theorem.

THEOREM 1 (The Principle of Monotonicity) *Let θ be a positive (respectively, negative) subformula of formula ϕ, let $\theta \rightarrow \rho$ (respectively, $\rho \rightarrow \theta$) be a formula, and let ϕ' be obtained from ϕ by substituting ρ for θ in ϕ. Then from $\theta \rightarrow \rho$ (respectively, $\rho \rightarrow \theta$), $\phi \rightarrow \phi'$ can be inferred.*

proof: For a proof of the Principle of Monotonicity consult [1], Theorem 2105, Substitutivity of Implication.

COROLLARY 2 *Let θ be a positive (respectively, negative) subformula of formula ϕ, and let ϕ' be obtained from ϕ by substituting $\top$ (respectively, $\bot$) for θ in ϕ. Then from ϕ, ϕ' can be inferred.*

3 Fluted Formulas

This section defines the syntax of fluted formulas. In this definition and throughout the paper variables will be retained to more clearly show the relation of FL to FO. Let L be a lexicon, i.e., a set of nonlogical predicates. Let $X_m := \{x_1, \ldots, x_m\}$ be an ordered set of m variables where $m \geq 0$. An *atomic fluted formula of L over X_m* is $Rx_{m-n+1} \cdots x_m$, where $R \in L$ and $ar(R) = n \leq m$. The set of all atomic fluted formulas of L over X_m will be denoted $Af_L(X_m)$. Define $Af_L(X_0) := \{\top\}$, where $\top$ is verum. $\bot := \neg\top$ is falsum.

A *fluted formula of L over X_m* is defined inductively. The *quantifier rank (qr)* is defined simultaneously.

1. An atomic fluted formula of L over X_m is a fluted formula of L over X_m with quantifier rank 0.

2. If ψ is a fluted formula of L over X_m with quantifier rank r, then $\neg\psi$ is a fluted formula of L over X_m with quantifier rank r.

3. If ψ and ϕ are fluted formulas of L over X_m with quantifier ranks r_ϕ and r_ψ, respectively, then $\psi \wedge \phi$, $\psi \vee \phi$, and $\psi \rightarrow \phi$ are fluted formulas of L over X_m with quantifier rank $max(r_\phi, r_\psi)$.

4. If ψ is a fluted formula of L over X_{m+1} with quantifier rank r, then $\exists x_{m+1}\psi$ and $\forall x_{m+1}\psi$ are fluted formulas of L over X_m with quantifier rank $r + 1$.

These fluted formulas will be referred to as *standard* fluted formulas. In addition, any first-order formula that can be transformed into a standard fluted formula by a consistent renaming of variables (both bound and free) is defined to be a fluted formula. No other formula is a fluted formula. Note that ψ is a standard fluted formula *over* X_m iff its free variables are exactly $\{x_i, x_{i+1}, \ldots, x_m\}$ for some $i, 1 \leq i \leq m+1$, and its bound variables are exactly $\{x_{m+1}, \ldots, x_{m+r}\}$ for $r = qr(\psi)$.

The semantics of the fluted formulas of L coincides with the standard semantics of first-order predicate logic. Also the semantic consequence relation is that of first-order logic, defined: $\psi \models \phi :\Leftrightarrow$ for all L-structures $\mathcal{A}$, for all assignments $\boldsymbol{a} \in A^\omega$, $\mathcal{A}, \boldsymbol{a} \models \psi$ only if $\mathcal{A}, \boldsymbol{a} \models \phi$.

4 Fluted constituents

A constituent is a generalization to FO of the *minterm* of Boolean logic. In Boolean logic it is proved that any Boolean formula is equivalent to a disjunction of minterms (i.e., *minimal conjunctions*). Similarly, in FO it is proved that any first-order formula is equivalent to a disjunction of constituents. For a review of constituent theory applied to FO see Rantala [9].

Constituents are especially useful in fluted logic since they have a simple representation as labeled trees. (See Section 5.) In this form, deciding consistency of a constituent is simple, requiring only inspection.

This section reviews the main results of Hintikka's constituent theory applied to fluted logic.

Let Φ be any set of prime formulas. A conjunction in which for each $\rho \in \Phi$ either ρ or $\neg\rho$ (but not both) occurs as a conjunct is a minimal conjunction over Φ. The set of minimal conjunctions over Φ will be denoted $\Delta\Phi$. It is well-known from Boolean logic that if $\Delta\Phi = \{\theta_1, \ldots, \theta_l\}$, and ψ is any Boolean combination of formulas of Φ, then the following are tautologies.

1. $\neg(\theta_i \wedge \theta_j)$, for $i \neq j$
2. $\theta_1 \vee \cdots \vee \theta_l$
3. either $\theta_i \rightarrow \psi$ or $\theta_i \rightarrow \neg\psi$, for $1 \leq i \leq l$

Of particular interest is $\Phi = Af_L(X_m)$, since this forms the basis case in the definition of fluted constituents.

The results for Boolean logic can be extended to quantificational fluted logic. First define the following operations on sets of formulas. Let Θ be a set of formulas.

$$\neg\Theta := \{\neg\theta : \theta \in \Theta\}$$

$$\exists x\Theta := \{\exists x\theta : \theta \in \Theta\}$$

$$\forall x\Theta := \{\forall x\theta : \theta \in \Theta\}$$

Now fluted constituents are defined inductively as follows.

basis: $\Gamma_L^{(0)}(X_m) := \Delta Af_L(X_m)$

induction: $\Gamma_L^{(i+1)}(X_m) := \{\theta \wedge \bigwedge \exists x_{m+1}\Theta \wedge \forall x_{m+1} \bigvee \Theta : (\theta \in \Delta Af_L(X_m)) \wedge (\emptyset \neq \Theta \subseteq \Gamma_L^{(i)}(X_{m+1}))\}$

A formula $\phi \in \Gamma_L^{(h)}(X_m)$ is a *fluted constituent of L of height h over the variables X_m*. If $m = 0$, then ϕ is a *fluted constituent sentence*. As defined here, height is synonymous with quantifier rank. Height is used to suggest a tree representation of the constituent. (Note that in the literature relating to constituent theory, the term 'depth' is so used.)

Now the main results of constituent theory, applied to fluted logic, can be given.

THEOREM 3 *1.* (Incompatibility Property) *If ϕ and ψ are fluted constituents of L of height h over the variables X_m, and $\phi \neq \psi$, then $\phi \wedge \psi$ is inconsistent.*

2. (Exhaustiveness Property) *The disjunction of all fluted constituents of L of height h over the variables X_m is logically valid.*

proof: It suffices to observe first that $\theta \wedge \bigwedge \exists x_{m+1}\Theta \wedge \forall x_{m+1} \bigvee \Theta$ in the definition of $\Gamma_L^{(i+1)}(X_m)$ is equivalent to $\theta \wedge \bigwedge \exists x_{m+1}\Theta \wedge \bigwedge \neg\exists x_{m+1}(\Gamma_L^{(i)}(X_{m+1}) - \Theta)$, and then to observe that this is a minimal conjunction over a set of prime formulas.

It follows from the Exhaustiveness Property that if $\mathcal{A}$ is any L-structure, $\mathcal{A} \models \bigvee \Gamma_L^{(n)}(X_0)$. Further, because of the Incompatibility Property, $\mathcal{A} \models \phi$ for exactly one constituent $\phi \in \Gamma_L^{(n)}(X_0)$. Thus there exists a many-one relation from L-structures to $\Gamma_L^{(n)}(X_0)$ for each n. This observation can be generalized to $\Gamma_L^{(n)}(X_k)$. It can be shown that consistent countable sets of constituents $\{\phi^{(n)} : \phi^{(n)} \in \Gamma_L^{(n)}(X_k) \wedge n \in \omega\}$ correspond to *k-types* [9].

It was remarked that any first-order formula is equivalent to a disjunction of first-order constituents. The disjunction of constituents to which a formula ψ is logically equivalent is called a *distributive normal form of* ψ. The analogous situation holds in fluted logic. The constituents in a distributive normal form of ψ are a subset of the constituents of some lexicon L and height h. In this case the distributive normal form of ψ is denoted $\bigvee \Gamma_L^{(h)}(X_k)(\psi)$, or simply $\bigvee \Gamma_\psi$ when no confusion can result. The disjuncts are called the *constituents of* ψ *of lexicon* L *and height* h. Notice that if ψ is a fluted formula over X_k, with quantifier rank h and lexicon L, then ψ has constituents $\Gamma_{L'}^{(n)}(X_k)(\psi)$ for every $n \geq h$ and $L' \supseteq L$.

THEOREM 4 *Let* ψ *be a standard fluted formula of* L *over* X_m*, where* $qr(\psi) = r$*. Then for every* $n \geq r$*,* ψ *is logically equivalent to the disjunction of constituents* $\Gamma_\psi \subseteq \Gamma_L^{(n)}(X_m)$*.*

proof: The proof is by induction on the complexity of ψ.

COROLLARY 5 *Let* ψ *be a standard fluted formula of* L *over* X_m*, where* $qr(\psi) = r$*. Let* ϕ *be a constituent of* L *of height* $n \geq r$ *over* X_m*. Then either* $\phi \rightarrow \psi$ *or* $\phi \rightarrow \neg\psi$ *is logically valid.*

proof: The corollary follows immediately from Theorem 4 and the Incompatibility Property of constituents.

5 Tree Representation

A fluted constituent sentence can be represented as a tree. This section describes that representation.

Let $\mathbf{P}^*$ be the set of finite strings over $\mathbf{P}$, the positive integers. String concatenation is denoted by juxtaposition. The empty string is ε.

A subset $\mathcal{T} \subseteq \mathbf{P}^*$ is a *tree domain* if

1. $\varepsilon \in \mathcal{T}$, and

2. if $\sigma i \in \mathcal{T}$, where $\sigma \in \mathbf{P}^*$ and $i \in \mathbf{P}$, then

   (a) $\sigma j \in \mathcal{T}$ for $0 < j < i$, and

   (b) $\sigma \in \mathcal{T}$.

Define the *height of* $\sigma \in \mathcal{T}$, $h(\sigma) :=$ the length of string σ. For all $\sigma, \tau \in \mathbf{P}^*, i \in \mathbf{P}$, if $\sigma i \tau \in \mathcal{T}$ then $\sigma i \tau$ is a *descendant of* σ and σi is an *immediate descendant of* σ. Define $w(\sigma) :=$ the number of immediate descendants of σ. Thus $\sigma 1, \sigma 2, \ldots, \sigma w(\sigma)$ are the immediate descendants of σ. If $w(\sigma) = 0$, then σ is *terminal* in $\mathcal{T}$. If all terminal elements of $\mathcal{T}$ have the same height, then $\mathcal{T}$ is *balanced*. In this case, $h(\mathcal{T}) := h(\sigma)$, where σ is any terminal element in $\mathcal{T}$. Define the *depth of* $\sigma \in \mathcal{T}$, $d(\sigma) := h(\mathcal{T}) - h(\sigma)$. If $0 < h(\sigma) < h(\mathcal{T})$, then σ is *internal* in $\mathcal{T}$. An element σ together with all of its descendants is defined to be the *subtree rooted on* σ, and is denoted $(\sigma]$.

Let $\mathcal{T}$ be a balanced tree domain. A *labeled tree domain* $\mathcal{T}_L$ is defined to be $\mathcal{T}$ with a formula $\theta_\sigma \in \Delta A f_L(X_{h(\sigma)})$ associated with each $\sigma \in \mathcal{T}$. For convenience, if $P\boldsymbol{x}$ is a conjunct in θ_σ, we write $P \in \theta_\sigma$. The labeled subtree of $\mathcal{T}_L$ rooted on σ will be denoted $(\theta_\sigma]$. The subtree $(\theta_\sigma]$ is given the following interpretation.

1. If σ is terminal, then $(\theta_\sigma]$ denotes θ_σ.

2. If σ is nonterminal with height k, then $(\theta_\sigma]$ denotes $\theta_\sigma \wedge \exists x_{k+1}(\theta_{\sigma 1}] \wedge \cdots \wedge \exists x_{k+1}(\theta_{\sigma w(\sigma)}] \wedge \forall x_{k+1}((\theta_{\sigma 1}] \vee \cdots \vee (\theta_{\sigma w(\sigma)}])$.

Thus the formula denoted by $(\theta_\sigma]$ is a fluted constituent of L of height $h(\mathcal{T}) - h(\sigma)$ over the variables $X_{h(\sigma)}$. If $h(\sigma) = 0$, the formula denoted by $(\theta_\sigma]$ is a fluted constituent sentence. If $\theta_\varepsilon = \neg\top$, then $\mathcal{T}_L$ is *trivial*.

In the sequel, all tree domains will be nontrivial labeled balanced tree domains. Moreover, $(\theta_\sigma]$ will not be distinguished from the formula it denotes.

There is an easy test for inconsistency of constituents, based on omission of variables. If ϕ is a fluted constituent, then $\phi^{[-k]}$ is defined to be ϕ with the last k variables omitted, and $\phi_{[-k]}$ is defined to be ϕ with the first k variables omitted. Here omission of a variable is accomplished by removing all atomic formulas in which that variable occurs, as well as the quantifier, if any, associated with that variable, and any connectives that thereby become idle. Semantically, the L- structures that satisfy a consistent constituent ϕ form a subclass of the L-structures that satisfy $\phi^{[-1]}$. The same result holds for $\phi_{[-1]}$. But as the following theorem shows, the superclass is the same in both cases.

THEOREM 6 *A constituent sentence ϕ is inconsistent unless $\phi^{[-1]}$ and $\phi_{[-1]}$ are equivalent.*

proof: Since ϕ is a constituent sentence, $\phi \to \phi^{[-1]}$ and $\phi \to \phi_{[-1]}$ by the Principle of Monotonicity. Hence $\phi \to (\phi^{[-1]} \wedge \phi_{[-1]})$. Moreover, $\phi^{[-1]}$ and $\phi_{[-1]}$ are constituent sentences of the same height. It follows from the Incompatibility Property that either $\phi^{[-1]}$ and $\phi_{[-1]}$ are equivalent (i.e., identical up to possible repetition of constituents, order of conjunction and disjunction, and change of variable), or ϕ is inconsistent. Hence the theorem follows.

Note that deciding whether ϕ satisfies this condition requires only inspection of (the tree representation of) ϕ. For this reason, if a constituent fails to satisfy the condition of Theorem 6, the constituent is said to be *trivially inconsistent* (cf. Hintikka [2, 3]). Of course Theorem 6 holds for FO as well as FL. But in contrast to the state of affairs in FO, in FL the converse of Theorem 6 holds (Purdy [8]). That is, the condition of Theorem 6 is both necessary and sufficient for consistency. To prove the converse, it is shown that the tree representation of a fluted constituent sentence satisfying the condition of Theorem 6 can be used to build a model of that constituent. (See below.)

The corresponding elements of $\phi^{[-1]}$ and $\phi_{[-1]}$ are related under the relation $\searrow$. For any fluted formula ϕ, define $\phi^{\dagger}$ to be ϕ with each variable (both bound and free) x_i replaced by x_{i-1}. Now $\searrow$ is defined:

1. for $1 \leq i \leq w(\varepsilon) : i \searrow \varepsilon$;

2. for $\sigma i, \tau j \in \mathcal{T} : \sigma i \searrow \tau j$ iff $\sigma \searrow \tau$ and $((\theta_{\sigma i}]_{[-1]})^{\dagger} = (\theta_{\tau j}]^{[-1]}$.

Let $\mathcal{T}_L$ be a tree representing a constituent sentence. Define a system of weights $\{n_\sigma : \sigma \in \mathcal{T}_L\}$ as follows.

1. For each $\sigma \in \mathcal{T}_L : n_\sigma \geq 1$.

2. If $\sigma i \searrow \tau j$, then $\Sigma_{\sigma l \searrow \tau j} n_{\sigma l} = \Sigma_{\sigma i \searrow \tau k} n_{\tau k}$.

There are infinitely many solutions for the system of weights $\{n_\sigma : \sigma \in \mathcal{T}_L\}$.

Each solution determines a model M for $\mathcal{T}_L$ as follows. $M := (A, F)$, where $A = \boldsymbol{n}_\varepsilon = \{0, 1, \ldots, n_\varepsilon - 1\}$, and a system of valuations $\mathcal{V} := \{V_\sigma : \sigma \in \mathcal{T}\}$ is defined:

basis: $V_1 := \{0, 1, \ldots, n_1 - 1\}, V_2 := \{n_1, \ldots, n_2 - 1\}, \ldots, V_{w(\varepsilon)} := \{n_{w(\varepsilon)-1}, \ldots, n_\varepsilon - 1\}$.

induction: Suppose $\sigma \searrow \tau$. Then construct $V_{\sigma i}$ so that $V_{\sigma i} \subseteq (V_\sigma \times A) \cap (A \times \{V_{\tau j} : \sigma i \searrow \tau j\})$ and $V_{\sigma i}$ is assigned $n_{\sigma i}$ such strings.

For $P \in L, F(P) := \{V_{\sigma[-(h(\sigma)-ar(P))]} : P \in \theta_\sigma\}$. That M is a model of $\mathcal{T}_L$ is shown by induction on depth of σ in $\mathcal{T}$. (for details, see Purdy [7].)

Alternatively, duplicate subconstituent $[\theta_\sigma)$ $n_\sigma - 1$ times so that there are a total of n_σ subconstituents identical to $[\theta_\sigma)$. Now $\{[\theta_{\sigma i})^{\dagger}_{[-1]} : 1 \leq i \leq w(\sigma)\}$ and $\{[\theta_{\tau j})^{[-1]} : 1 \leq i \leq w(\tau)\}$ are equal *as multisets*. Define a new relation $\curvearrowright$ as follows. Arbitrarily assign $[\sigma i) \curvearrowright [\tau j)$, in 1:1 fashion, subject to $[\theta_{\sigma i})^{\dagger}_{[-1]} = [\theta_{\tau j})^{[-1]}$. It follows from the definition of $\{n_\sigma : \sigma \in \mathcal{T}_L\}$ that there are exactly enough subconstituents $[\theta_{\sigma i})$ and $[\theta_{\tau j})$ so that for each σi, there exists a unique τj such that $\sigma i \curvearrowright \tau j$ and conversely. Now the model of $\mathcal{T}_L$ can be constructed so that $V_{\sigma i} = (V_\sigma \times A) \cap (A \times V_{\tau j})$. Such a constituent with duplicated subconstituents will be referred to as a *flattened* constituent.

6 Fluted structures

Let $\mathcal{A} = (A, F)$ be a first-order structure, where A is a nonempty set and for each $P \in L : F(P) \subseteq A^{ar(P)}$. Let $n = ar(L)$. Define the *natural frame* for $\mathcal{A}$: $\mathcal{F}^{(n)} := (S^{(n)}, R)$, where $S^{(n)} = \bigcup_{i=0}^{n} A^i$ (the state set) and $R = \{(\sigma, \sigma a) : a \in A \wedge h(\sigma) < n\}$ (the accessibility relation). $\mathcal{F}^{(n)}$ is a transition system. As with constituents, define the *height of* σ: $h(\sigma) :=$ the length of the string σ. For each $\sigma \in S^{(n)} : \theta_\sigma \in \Delta A f_L(X_{h(\sigma)})$ such that $\mathcal{A}, \sigma \models \theta_\sigma$. This labeled transition system, $M = (\mathcal{F}^{(n)}, \theta_\sigma)_{\sigma \in S^{(n)}}$, will be called a *fluted structure*. $(\mathcal{F}^{(n)}, \theta_\sigma)_{\sigma \in S^{(n)}} \models \phi$ means that at some state σ, ϕ is satisfied. If ϕ is satisfied at a particular state σ, we write $\sigma \models \phi$. Note that it is essential that θ_σ be a fluted formula, since if $P \in \theta_\sigma$ and permution π were applied to the arguments of P (through the use of variables), P would effectively be moved to the label of a different state $\pi(\sigma)$.

Remarks: Let $\mathcal{T}_L$ be a constituent of height l.

1. If $ar(P) < h(\sigma)$ and $h(\sigma) \leq l$, $P \in \theta_\sigma \rightarrow P \in \theta_{\sigma_{[-1]}}$.

2. If $h(\sigma) < l$, $P \in \theta_\sigma \rightarrow P \in \theta_{a\sigma}$ for all $a \in A$.

3. As a result, $(\mathcal{F}^{(n)}, \theta_\sigma)_{\sigma \in S^{(n)}}$ and $(\mathcal{F}^{(n+1)}, \theta_\sigma)_{\sigma \in S^{(n+1)}}$ are equivalent in the following sense. $S^{(n)} = \bigcup_{i=0}^{n} A^i$ and $S^{(n+1)} = \bigcup_{i=0}^{n+1} A^i$. That is, $S^{(n+1)} = S^{(n)} \cup A^{n+1} = S^{(n)} \cup \{\sigma a : \sigma \in S^{(n)} \wedge h(\sigma) = n \wedge a \in A\}$. Further, $\theta_{\sigma a} = \theta_\sigma$ for all σ at height n and for some $a \in A$ such that the suffix of length n of σa is equal to σ. Thus $(\mathcal{F}^{(n)}, \theta_\sigma)_{\sigma \in S^{(n)}}$ uniquely determines $(\mathcal{F}^{(n+1)}, \theta_\sigma)_{\sigma \in S^{(n+1)}}$. By the Principle of Monotonicity, the converse holds as well. Inductively, in the same sense, $(\mathcal{F}^{(n)}, \theta_\sigma)_{\sigma \in S^{(n)}}$ and $(\mathcal{F}^{(n+k)}, \theta_\sigma)_{\sigma \in S^{(n+k)}}$ are equivalent for all $k \geq 0$.

4. $(\mathcal{F}^{(n+k)}, \theta_\sigma)_{\sigma \in S^{(n+k)}}$ corresponds to a unique constituent $\gamma \in \Gamma_L^{(n+k)}()$. Indeed, if the state names are ignored, the frame is identical to some flattened tree representation of γ. Moreover, if $\sigma \models \phi$, it follows that $(\theta_\sigma] \in \Gamma_L^{(n+k-h(\sigma))}(X_{h(\sigma)})$ is a constituent of ϕ.

If $h(\sigma) = n, P \in \theta_\sigma$, and $R\tau\sigma$, then $\sigma = \tau a$ for some $a \in A$, whence $\tau \models \exists x_n P x_n$. Similarly, if $P \in \theta_\sigma$ for every σ such that $R\tau\sigma$, then $\tau \models \forall x_n P x_n$. In general, suppose $\sigma \models \psi$ and $R\tau\sigma$. Then $\tau \models \exists x_n \psi$. If $\sigma \models \psi$ for every σ such that $R\tau\sigma$, then $\tau \models \forall x_n \psi$. Every fluted structure defines a frame $\mathcal{F}^{(n+k)}$ for each $k \geq 0$ and thence a unique constituent in $\Gamma_L^{(n+k)}$.

In summary, let

1. $\boldsymbol{A}$:= first-order L-structures with domain A;
2. $\boldsymbol{M}$:= fluted L-structures with domain A;
3. $\boldsymbol{F}$:= natural frames for first-order L-structures with domain A;
4. $\Gamma_L^{(n)}()$:= fluted constituent sentences of height n.

Then

1. $\boldsymbol{A} \to \boldsymbol{F}$ is a 1:1 map;
2. $\boldsymbol{A} \to \boldsymbol{M}$ is a 1:1 map;
3. $\boldsymbol{M} \to \Gamma_L^{(n)}()$ is a many:1 map.

That is, for each first-order structure with domain A, there exists a unique natural frame with domain A and a unique fluted structure with domain A. Moreover, for each fluted structure, there exists a unique fluted constituent (for each height $n \geq ar(L)$) that characterizes that structure.

7 Expressiveness of fluted logic

Again, let $n = ar(L)$. Suppose $M_1, M_2 \in \boldsymbol{M}$ and ϕ is any fluted sentence of quantifier rank $\leq n$. Let $M_1 \models \gamma_1 \in \Gamma_L^{(n)}()$ and $M_2 \models \gamma_2 \in \Gamma_L^{(n)}()$. Then ϕ cannot distinguish between M_1 and M_2 if $\gamma_1 = \gamma_2$, since if $M_1 \models \phi$, $M_1 \models \phi \wedge \gamma_1$ which implies $\models \gamma_1 \to \phi$, and so if $M_2 \models \gamma_1$ then $M_2 \models \gamma_1 \wedge (\gamma_1 \to \phi)$ whence $M_2 \models \phi$. Similarly, if $M_1 \models \neg\phi$, then $M_2 \models \neg\phi$. Of course, if $\gamma_1 \neq \gamma_2$, then both γ_1 and γ_2 can distinguish between M_1 and M_2. This yields the theorem

THEOREM 7 *If $M_1, M_2 \in \boldsymbol{M}$ such that $M_1 \models \gamma_1 \in \Gamma_L^{(n)}()$ and $M_2 \models \gamma_2 \in \Gamma_L^{(n)}()$, then there exists a fluted sentence ϕ of quantifier rank $\leq n$ which can distinguish between M_1 and M_2 iff $\gamma_1 \neq \gamma_2$.*

This result easily generalizes to fluted formulas and to arbitrary domains A_1 and A_2.

Of course, Theorem 7 holds for the first-order case as well, but the simplicity of constructing a counterexample is lacking. Nonetheless, Theorem 7 does have utility beyond FL, as shown by Example 4 below.

Now several examples are presented. These examples establish important limits on the expressiveness of fluted formulas.

Example 1. Define model M_1: domain $A := \{a, b\}$, $F(P) := A$, $F(Q) := \{ab, ba\}$. Relation Q is symmetric. Define model M_2: domain $A := \{a, b\}$, $F(P) := A$, $F(Q) := \{ab, bb\}$. Relation Q is not symmetric. But M_1 and M_2 satisfy the same constituent, which can easily be seen by constructing the fluted structures, the flattened constituents, and the constituents. From this counterexample, it follows that no fluted formula can express symmetry.

Example 2. Define model M_1: domain $A := \{a, b\}$, $F(P) := A$, $F(Q) := \{aa, bb\}$. Relation Q is reflexive. Define model M_2: domain $A := \{a, b\}$, $F(P) := A$, $F(Q) := \{ab, ba\}$. Relation Q is not reflexive. But M_1 and M_2 satisfy the same constituent, which again can easily be seen by constructing the fluted structures, the flattened constituents, and the constituents. From this counterexample, it follows that no fluted formula can express reflexivity.

Example 3. Define model M_1: domain $A := \{a, b, c\}$, $F(P) := A$, $F(Q) := \{ab, bc, ac\}$. Relation Q is transitive. Define model M_2: domain $A := \{a, b, c\}$, $F(P) := A$, $F(Q) := \{aa, ab, bc\}$. Relation Q is not transitive. But M_1 and M_2 satisfy the same constituent. Again, this can easily be seen by constructing the fluted structures, the flattened constituents, and the constituents. From this counterexample, it follows that no fluted formula can express transitivity.

Example 4. In this example, the fragment FO^2, first-order logic in which only two distinct variable symbols can occur, is considered. The natural frame is constructed as before, but θ_σ is an element of $\Delta At2_L$, the set of minimal conjunctions of atomic FO^2 formulas. Define model M_1: domain $A := \{a, b, c, d\}$, $F(P) := A$, $F(Q) := \{aa, ab, ba, bb, cc, dd\}$. Relation Q is transitive. Define model M_2: domain

$A := \{a, b, c, d\}$, $F(P) := A$, $F(Q) := \{aa, ab, ba, bb, bc, cb, cc, dd\}$. Relation Q is not transitive. But M_1 and M_2 satisfy the same constituent. This can easily be seen by constructing the FO^2 structures, the flattened constituents, and the constituents. From this counterexample, it follows that no FO^2 formula can express transitivity.

In a similar manner, it can be shown that antisymmetry is beyond the reach of fluted logic. These examples show that it is not possible to specify a partial order relation or an equivalence relation with either fluted logic or FO^2, and therefore that these fragments are not useful for reasoning about mathematical entities. But fluted logic does excel in construal of syllogistic, and its extension to polyadic relations. That is, fluted logic is an excellent natural language reasoning environment. In syllogistic, the $is - a$ relation is basic. For example: **Every maple is-a tree.** is construed $\forall x(maple(x) \rightarrow tree(x))$, or in variable-free form, $\forall(maple \rightarrow tree)$. Again, **No man is-a island.** (i.e., **not: some man is-a island**) is interpreted $\neg\exists x(man(x) \wedge island(x))$, or $\neg\exists(man \wedge island)$. This is extended to polyadic relations as exemplified by the $of - a$ relation: **Every cat is-a companion of-a human.**, which becomes $\forall x(cat(x) \rightarrow \exists y(companion(x, y) \wedge human(y)))$ The $of - a$ relation is typical of relations that take a subject and an object, such as prepositions and transitive verbs. Relations of higher arity, such as verbs that take a subject, a direct object, and an indirect object (for example, **give**) are treated similarly. Example: **A teacher gives a quiz to a student.** becomes $\exists(teacher \wedge \exists(quiz \wedge \exists(student \wedge give)))$.

8 Discussion

The principal result of this paper is that each first-order structure determines a natural transition system (called the 'natural frame' for that structure). This transition system, appropriately labeled, in turn determines a fluted constituent sentence characteristic of that structure. Two structures can be distinguished by fluted formulas iff the constituents characteristic of the two structures differ. A similar conclusion is reached for FO^2. It easily follows from this result that as fragments of FO, FL and FO^2 suffer significant deficits in expressiveness. These deficits make them unsuitable for specifying and reasoning about most mathematical entities.

However, FL does have the expressiveness of syllogistic and can easily extend syllogistic to polyadic relations. Construal of natural language in FL has a naturalness and an intuitive nature that FO and most of its fragments lack.

Moreover, reasoning in FL is decidable ([8]). At least one decision algorithm exists for FL ([11]).

Consequently, FL is particularly well-suited for construing natural language reasoning.

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