Strong Bisimulations

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(R1 \{ (A|B, D), (b.A|B, E) \} is a strong bisimulation, because:
- For (A|B, D):
  - A|B \xrightarrow{a} b.A|B can be matched by D \xrightarrow{a} E, and (b.A|B, E) \in R_1
  - A|B \xrightarrow{c} A|B can be matched by D \xrightarrow{c} D, and (A|B, D) \in R_1
  - D \xrightarrow{a} E can be matched by A|B \xrightarrow{a} b.A|B, and (b.A|B, E) \in R_1
  - D \xrightarrow{c} D can be matched by A|B \xrightarrow{c} A|B, and (A|B, D) \in R_1
- For (b.A|B, E):
  - b.A|B \xrightarrow{b} A|B can be matched by E \xrightarrow{b} D, and (A|B, D) \in R_1
  - b.A|B \xrightarrow{a} b.A|B can be matched by E \xrightarrow{a} E, and (b.A|B, E) \in R_1
  - E \xrightarrow{c} E can be matched by b.A|B \xrightarrow{c} b.A|B, and (b.A|B, E) \in R_1
  - E \xrightarrow{b} D can be matched by b.A|B \xrightarrow{b} A|B, and (A|B, D) \in R_1

R_2 = \{ (A|B, F), (b.A|B, G) \} is not a strong bisimulation:
- (A|B, F) \in R_2, but:
  - F \xrightarrow{a} F, and only possible match is A|B \xrightarrow{a} b.A|B, and (b.A|B, F) \notin R_2

R_3 = \{ (A|B, F), (b.A|B, G), (b.A|B, F) \} is not a strong bisimulation:
- (b.A|B, F) \in R_3, but:
  - b.A|B \xrightarrow{b} A|B, and there is no Q' such that F \xrightarrow{b} Q'

Strong Bisimulations: A Non-Example

A \overset{\text{def}}{=} a.b.A \quad D \overset{\text{def}}{=} c.D + a.E \quad F \overset{\text{def}}{=} c.F + a.G + a.F
B \overset{\text{def}}{=} c.B \quad E \overset{\text{def}}{=} c.E + b.D \quad G \overset{\text{def}}{=} c.G + b.F

Strong Bisimulations: An Example

R_1 = \{ (A|B, D), (b.A|B, E) \} is a strong bisimulation, because:
- For (A|B, D):
  - A|B \xrightarrow{a} b.A|B can be matched by D \xrightarrow{a} E, and (b.A|B, E) \in R_1
  - A|B \xrightarrow{c} A|B can be matched by D \xrightarrow{c} D, and (A|B, D) \in R_1
  - D \xrightarrow{a} E can be matched by A|B \xrightarrow{a} b.A|B, and (b.A|B, E) \in R_1
  - D \xrightarrow{c} D can be matched by A|B \xrightarrow{c} A|B, and (A|B, D) \in R_1
- For (b.A|B, E):
  - b.A|B \xrightarrow{b} A|B can be matched by E \xrightarrow{b} D, and (A|B, D) \in R_1
  - b.A|B \xrightarrow{a} b.A|B can be matched by E \xrightarrow{a} E, and (b.A|B, E) \in R_1
  - E \xrightarrow{c} E can be matched by b.A|B \xrightarrow{c} b.A|B, and (b.A|B, E) \in R_1
  - E \xrightarrow{b} D can be matched by b.A|B \xrightarrow{b} A|B, and (A|B, D) \in R_1

Strong Bisimulations

Definition
A relation \( R \subseteq \text{Proc} \times \text{Proc} \) over the set of CCS processes is a (strong) bisimulation if and only if:

For all pairs \((P, Q) \in R\) and all actions \( \alpha \in \text{Act} \):
1. If \( P \overset{\alpha}{\rightarrow} P' \), then there exists \( Q' \) such that \( Q \overset{\alpha}{\rightarrow} Q' \) and \((P', Q') \in R\).
2. If \( Q \overset{\alpha}{\rightarrow} Q' \), then there exists \( P' \) such that \( P \overset{\alpha}{\rightarrow} P' \) and \((P', Q') \in R\).

A simple (trivial!) example:
- \( \emptyset \) is a strong bisimulation.
Bisimulation Equivalence

**Definition**

Two processes $P$ and $Q$ are (strongly) bisimilar (a.k.a. bisimulation equivalent) if and only if:

There exists some bisimulation $B$ such that $(P, Q) \in B$.

When $P$ and $Q$ are strongly bisimilar, we write $P \sim Q$.

**Recall bisimulation** $R_1 = \{(A|B, D), (b.A|B, E)\}$:

Thus, $A|B \sim D$ and $b.A|B \sim E$

**Theorem**

$\sim$ is an equivalence relation: For all processes $P, Q, R$:

- $P \sim P$ (Reflexivity)
- If $P \sim Q$, then $Q \sim P$ (Symmetry)
- If $P \sim Q$ and $Q \sim R$, then $P \sim R$ (Transitivity)

Other Important Properties about $\sim$

**Theorem**

Let $P$ and $Q$ be (finitely branching) CCS processes. Then $P \sim Q$ if and only if they satisfy exactly the same HML formulas:

$$P \sim Q \text{ if and only if } \{ \varphi \mid P \models \varphi \} = \{ \varphi \mid Q \models \varphi \}$$

Bisimulation Equivalence and HML

**Theorem**

Let $P$ and $Q$ be (finitely branching) CCS processes. Then $P \sim Q$ if and only if they satisfy exactly the same HML formulas:

$$P \sim Q \text{ if and only if } \{ \varphi \mid P \models \varphi \} = \{ \varphi \mid Q \models \varphi \}$$

- $b.A \not\sim F$, because $b.A \models \langle b \rangle \text{tt}$ but $F \not\models \langle b \rangle \text{tt}$
- $A \models \langle b \rangle \not\text{tt}$, because $A \models \langle a \rangle \not\text{tt}$ but $\not\models \langle a \rangle \text{tt}$