1. (24 points) Consider the Kripke structure $\mathcal{M} = \langle W, I, J \rangle$, where:

$$\begin{aligned} W &= \{x, y, t, w\} \\ I(q) &= \{y, t\} \\ I(r) &= \{x, t\} \\ I(s) &= \{x, w, y\} \\ J(Fay) &= \{(x,x),(y,x),(t,t),(t,w),(w,w)\} \\ J(Lem) &= \{(x,y),(x,w),(t,t),(w,x)\} \end{aligned}$$

   (a) (2 points) What is the value of $J(Lem \,|\, Fay)$?
   $\{(x,x),(x,w),(t,t),(t,w),(w,x)\}$

   (b) (2 points) What is the value of $J(Fay \,|\, Lem)$?
   $\{(x,y),(x,w),(y,y),(y,w),(t,t),(t,x),(w,x)\}$

   (c) (20 points) For each formula that follows, give the set of worlds in W in which it is true. You do not need to show your work.

      i. $r \supset q$
      $\{y, w, t\}$

      ii. $\neg(r \supset q)$
      $\{x\}$

      iii. Fay says $(r \supset q)$
      $\{t, w\}$

      iv. Lem says $(r \supset q)$
      $\{x, y, t\}$

      v. Lem says $\neg(r \supset q)$
      $\{y, w\}$

      vi. $Fay \,\&\, Lem$ says $(r \supset q)$
      $\{t\}$

      vii. Fay controls s
      $W = \{x, y, t, w\}$

      viii. $Lem \,|\, Fay \Rightarrow Fay$
      $\emptyset$

      ix. Lem says $(Lem \,|\, Fay \Rightarrow Fay)$
      $\{y\}$

      x. Lem controls $(Lem \,|\, Fay \Rightarrow Fay)$
      $\{x, w, t\}$

2. (12 points) Give a formal proof of the following derivable rule:

$$\frac{Q \text{ controls } \varphi \qquad P \text{ controls } (Q \text{ says } \varphi) \qquad P \,|\, Q \text{ says } \varphi}{\varphi}$$

In addition to the inference rules and definitions of Figure 3.1, you may use *without proof* any derived rule that appears in Chapter 3 of the text, including those listed in exercises.

1. Q controls φ	Assumption
2. P controls $(Q$ says $\varphi)$	Assumption
3. $P \mid Q$ says φ	Assumption
4. $P \mid Q$ says $\varphi \equiv P$ says Q says φ	Quoting
5. P says Q says φ	3,4 Equivalence
6. Q says φ	2,5 Controls
7. φ	1,6 Controls

3. (12 points) Prove that the following proposed inference rule is *sound*:

$$\frac{P \text{ says } (R \Rightarrow Q) \qquad \neg(R \Rightarrow Q)}{P \text{ says } \varphi}$$

That is, show that—for all Kripke structures $\mathcal{M}$, principals P, Q, R, and formulas φ—the following statement is true:

If $\mathcal{M} \models P$ says $(R \Rightarrow Q)$ and $\mathcal{M} \models \neg(R \Rightarrow Q)$, then $\mathcal{M} \models P$ says φ as well.

Proof: Consider an arbitrary Kripke structure $\mathcal{M} = \langle W, I, J \rangle$ such that

$$\mathcal{M} \models P \text{ says } (R \Rightarrow Q), \qquad \mathcal{M} \models \neg(R \Rightarrow Q).$$

By definition of $\models$,

$$\begin{aligned} W &= \mathcal{E}_{\mathcal{M}}[\![\neg(R \Rightarrow Q)]\!] \\ &= W - \mathcal{E}_{\mathcal{M}}[\![R \Rightarrow Q]\!], \end{aligned}$$

and thus $\mathcal{E}_{\mathcal{M}}[\![R \Rightarrow Q]\!] = \emptyset$.

Likewise, by definition of $\models$,

$$\begin{aligned} W &= \mathcal{E}_{\mathcal{M}}[\![P \text{ says } (R \Rightarrow Q)]\!] \\ &= \{w \mid J(P)w \subseteq \mathcal{E}_{\mathcal{M}}[\![R \Rightarrow Q]\!]\} \\ &= \{w \mid J(P)w \subseteq \emptyset\}. \end{aligned}$$

It follows that $J(P) = \emptyset$ necessarily.

Therefore, for any formula φ,

$$\begin{aligned} \mathcal{E}_{\mathcal{M}}[\![P \text{ says } (R \Rightarrow Q)]\!] &= \{w \mid J(P)w \subseteq \mathcal{E}_{\mathcal{M}}[\![\varphi]\!]\} \\ &= W, \end{aligned}$$

and thus $\mathcal{M} \models P \text{ says } \varphi$.

Because $\mathcal{M}$ was arbitrary, the rule is sound.

4. (12 points) Show that the following proposed inference rule is *not sound*, and therefore should not be added to the logic:

$$\frac{P \text{ says } (\varphi_1 \vee \varphi_2)}{(P \text{ says } \varphi_1) \vee (P \text{ says } \varphi_2)}$$

That is, give a particular Kripke structure $\mathcal{M}$, formulas φ_1 and φ_2, and principal P such that:

$$\mathcal{M} \models P \text{ says } (\varphi_1 \vee \varphi_2), \text{ but } \mathcal{M} \not\models (P \text{ says } \varphi_1) \vee (P \text{ says } \varphi_2).$$

For maximal credit, be sure to provide calculations to support your answer.

Answer: Let φ_1 be the formula q and φ_2 be r; let P be the principal Al.

Consider the model $\mathcal{M} = \langle W, I, J \rangle$, where:

$$\begin{aligned} W &= \{a, b\} \\ I(q) &= \{a\} \\ I(r) &= \{b\} \\ J(Al) &= \{(a,a), (a,b), (b,b)\} \end{aligned}$$

Note that $\mathcal{E}_\mathcal{M}[\![q \vee r]\!] = \mathcal{E}_\mathcal{M}[\![q]\!] \cup \mathcal{E}_\mathcal{M}[\![r]\!] = I(q) \cup I(r) = \{a, b\}$. In addition, we have

$$J(Al)a = \{a, b\}, \quad J(Al)b = \{b\}.$$

- $\mathcal{M} \models Al \text{ says } (q \vee r)$, because:

$$\begin{aligned} \mathcal{E}_\mathcal{M}[\![Al \text{ says } (q \vee r)]\!] &= \{w \mid J(Al)w \subseteq \mathcal{E}_\mathcal{M}[\![q \vee r]\!]\} \\ &= \{w \mid J(Al)w \subseteq \{a, b\}\} \\ &= \{a, b\} = W. \end{aligned}$$

- In contrast, $\mathcal{M} \not\models (Al \text{ says } q) \vee (Al \text{ says } r)$, because:

$$\begin{aligned} \mathcal{E}_\mathcal{M}[\![(Al \text{ says } q) \vee (Al \text{ says } r)]\!] &= \{w \mid J(Al)w \subseteq \mathcal{E}_\mathcal{M}[\![q]\!]\} \cup \{w \mid J(Al)w \subseteq \mathcal{E}_\mathcal{M}[\![r]\!]\} \\ &= \{w \mid J(Al)w \subseteq \{a\}\} \cup \{w \mid J(Al)w \subseteq \{b\}\} \\ &= \emptyset \cup \{b\} \\ &= \{b\} \neq W. \end{aligned}$$