CSE/CIS 607 Mathematical Basis of Computing

Assessment Exam

January 15, 2015

INSTRUCTIONS. Do all of the questions. Put your answers on the test paper. If you find yourself needing extra space, your are probably not taking a good approach.

NAME (print): ________________________________

SIGNATURE ________________________________
**Question 1a:** How many elements are there in a set $A \times B$, where $A$ has 5 elements and $B$ has 6 elements?

**Answer:**

$$\|A \times B\| = \|A\| \cdot \|B\| = 5 \cdot 6 = 30$$

**Question 1b:** How many dyadic, sometimes called binary, relations from a set $A$ with $m$ elements to a set $B$ with $n$ elements are there?

**Answer:**

$$2^{m \cdot n}$$
**Question 2:** Let $A$ be a set with exactly $m$ elements and $B$ be a set with exactly $n$ elements, where $m \leq n$. How many one-to-one, i.e. injective, functions are there from $A$ to $B$?

**Answer:** The number of such functions is the number of possible ranges for such a function time the number of functions mapping into each range:

\[
\binom{n}{m} m! = \frac{n}{(n - m)!}
\]
**Question 3:** Consider:

1. For every slithy tove, if that slithy tove gyres then Bob gymbols.
2. If every slithy tove gyres, then Bob gymbols.

Does (2) logically follow from (1)? Does (1) logically follow from (2)? Explain.

**Answer:** For a statement $\varphi_2$ to logically follow from a statement $\varphi_1$ it must be that in every situation in where $\varphi_1$ is true, $\varphi_2$ is also true. Equivalently (and contrapositively), in every situation where $\varphi_2$ is false, $\varphi_1$ is false. Applying this principle to the current example we will inquire into whether (2) follows from (1) by supposing that we have a situation where (2) is false, and see whether (1) must also be false, a do a similar inquiry to see whether (1) follows from (2).

Suppose (2) is false. Then it must be that every slithyTove gyres, but Bob doesn't gymbol. Now consider any slithyTove. Let's call the slithyTove, Dubya. Dubya gyres because every slithyTove gyres. Therefore the statement

If Dubya gyres, then Bob gymbols

is false. This instance of (1) is false, so (1) is false. In other words, Dubya is a counterexample to (1). We have just seen that the conditions that make (2) false also make (1) false. So (2) does follow from (1) after all.

Does (1) follow from (2)? Suppose (1) is false. Then there must be a counterexample to (1). So there is some slithyTove (lets call it Jeb) such that it is false that if Jeb gyres then Bob gymbols. In other words, Jeb gyres, but Bob doesn't gymbol. Well, even as Jeb gyres, we will suppose we are in a situation where there is another slithy Tove, Mitt, that doesn't gyre. So not every slithy Tove gyres in that situation. Then in that situation, (2) is true. So conditions that would make (1) false could still allow (2) to be true. Therefore, (1) does not follow from (2).
**Question 4:** Let $A$ be a set. The successor of set $A$ is the set $A \cup \{A\}$. Denote the successor of a set $S$ by $S'$. For example, $\emptyset' = \{\emptyset\}$, where $\emptyset$ is the empty set. Let

1. $\emptyset^{(0)} = \emptyset.$
2. $\emptyset^{(n+1)} = (\emptyset^{(n)})'$

**Part a)** List the elements in $\emptyset^{(3)}$.

**Answer:**

1. $\emptyset^{(1)} = (\emptyset^{(0)})' = \emptyset' = \emptyset \cup \{\emptyset\} = \emptyset$
2. $\emptyset^{(2)} = (\emptyset^{(1)})' = \emptyset' = \emptyset \cup \{\emptyset\} = \emptyset, \{\emptyset\}$
3. $\emptyset^{(3)} = (\emptyset^{(2)})' = \emptyset, \{\emptyset\}' = \emptyset, \{\emptyset\} \cup \{\emptyset, \{\emptyset\}\} = \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$

**Part b)** Use mathematical induction to prove

For every $n$, if $x \in \emptyset^{(n)}$ then $x$ is a subset of $\emptyset^{(n)}$.

**Answer:**

*Base $(n = 0)$:* We must prove:

If $x \in \emptyset^{(0)}$ then $x$ is a subset of $\emptyset^{(0)}$.

i.e.

If $x \in \emptyset$ then $x$ is a subset of $\emptyset$.

But the above statement is true, since the statement, $x \in \emptyset$, is false for any value of $x$.

*Induction:* We will prove:

For every $n$,

if

if $x \in \emptyset^{(n)}$ then $x$ is a subset of $\emptyset^{(n)}$
then
if \( x \in \emptyset^{(n+1)} \) then \( x \) is a subset of \( \emptyset^{(n+1)} \)

Consider an arbitrary but fixed value of \( n \). It remains to prove

\[
\text{if}
\begin{align*}
  & \text{if } x \in \emptyset^{(n)} \text{ then } x \text{ is a subset of } \emptyset^{(n)} \\
  \text{then}
  & \text{if } x \in \emptyset^{(n+1)} \text{ then } x \text{ is a subset of } \emptyset^{(n+1)}
\end{align*}
\]

Assume: if \( x \in \emptyset^{(n)} \) then \( x \) is a subset of \( \emptyset^{(n)} \). (This is the induction assumption). It remains to prove

\[
\text{if } x \in \emptyset^{(n+1)} \text{ then } x \text{ is a subset of } \emptyset^{(n+1)}.
\]

Assume (Assumption 2): \( x \in \emptyset^{(n+1)} \).

It remains to prove

\( x \) is a subset of \( \emptyset^{(n+1)} \).

Since \( \emptyset^{(n+1)} \) is the successor of \( \emptyset^{(n)} \), by assumption (2)

\[
x \in \emptyset^{(n)} \quad \text{or} \quad x = \emptyset^{(n)}
\]

By the induction assumption, and the fact that \( x = \emptyset^{(n)} \) implies \( x \subseteq \emptyset^{(n)} \),

\[
x \subseteq \emptyset^{(n)}
\]

By the definition of the successor of a set, and the transitivity of the subset relation, also known as set inclusion,

\[
x \subseteq \emptyset^{(n+1)}
\]

which was what remained to proved in the induction step.

By the principle of mathematical induction, it follows that

For every \( n \), if \( x \in \emptyset^{(n)} \) then \( x \) is a subset of \( \emptyset^{(n)} \).
Question 5:

Part 1: Let $R$ be a reflexive binary relation from a set $A$ to $A$. Prove that $R$ is transitive if, and only if, $R = R \circ R$.

Answer: Suppose $R$ is transitive and that $(x, y)$ is an arbitrary ordered pair in $R$. Since $R$ is reflexive, $(x, x) \in R$. Therefore, $(x, y) \in R \circ R$. Hence, $R \subseteq R \circ R$. Conversely, suppose $(x, y) \in R \circ R$. Then there is some $w$ such that $(x, w) \in R$ and $(w, y) \in R$. Since $R$ is transitive, $(x, y) \in R$.

Now suppose $R = R \circ R$ and that $(x, y) \in R$ and $(y, z) \in R$. So $(x, z) \in R \circ R$. Since $R = R \circ R$, $(x, z) \in R$. Therefore $R$ is transitive.

Part 2: Give an example of a transitive binary relation from a set $A$ to $A$ for which $R \neq R \circ R$.

Answer: Let $A = \{0, 1\}$ and $R = \{(0,1), A, A\}$. $R \circ R = \emptyset$. 
Appendix

**Definition 0.1** For any sets $A$ and $B$, the set $A \times B$, called the *Cartesian Product* of $A$ and $B$, is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$. In other words,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$  

**Example 0.1** Let $A = \{0, 1, 2\}$ and let $B = \{0, 1\}$. Then

$$A \times B = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$  

**Example 0.2** Let $\mathbb{N}$ be the set of non-negative integers. That is, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Then

$$\mathbb{N} \times \mathbb{N} = \{(m, n) \mid m \in \mathbb{N} \text{ and } n \in \mathbb{N}\}.$$  

**Definition 0.2** For any sets $A$ and $S$, $A$ is a *subset* of $S$ if, and only if, every member of $A$ is a member of $S$. We denote that $A$ is a subset of $S$ by $A \subseteq S$.

**Definition 0.3** For any set $S$, the *power set* of $S$, denoted by $P(S)$, is the set of all subsets of $S$. That is,

$$P(S) = \{B \mid B \subseteq S\}.$$  

**Note:** The definition of power set implies that for anything $x$, $x \in P(S)$ if, and only if, $x \subseteq S$.

**Definition 0.4** A *dyadic*, also called a *binary*, relation $R$ from set $A$ to set $B$ is a triple consisting of (1) a subset of $A \times B$, (2) $A$, and (3) $B$. A *dyadic*, also called a *binary* relation $R$ on a set $A$ is a *dyadic*, or *binary relation* $R$ from $A$ to set $A$. We shall call dyadic relations, simply, relations, when the context is clear.

**Definition 0.5** A relation $f$ from $A$ to $B$ is called a *function* from $A$ to $B$ if, and only if, for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Suppose $f$ is a function from $A$ to $B$. Given $a \in A$, the $b \in B$ for which $(a, b) \in f$ is denoted by $f(a)$. The expression $f : A \longrightarrow B$ means that $f$ is a function from $A$ to $B$.

**Definition 0.6** A function $f : A \longrightarrow B$ is called an *injection*, if and only if the following condition for $f$ is true: for all elements $x$ and $y$ of $A$ such that $x \neq y$: $f(x) \neq f(y)$. When a function is an injection, we also say that the function is *injective*, and we also say that the function is *one-to-one*. These phrases are equivalent ways of expressing the same thing.
Note: The definition says that $f : A \rightarrow B$ is injective if, and only if, for every two different inputs to $f$ we must get two different outputs. Equivalently, we cannot get the same output from $f$ from two different inputs. The condition for $f$ to be injective can be restated in the following equivalent form: for all elements $x$ and $y$ of $A$, if $f(x) = f(y)$, then $x = y$.

**Definition 0.7** A function $f : A \rightarrow B$ is called a surjection, if and only if, the following condition for $f$ is true: for each element $b$ of $B$, there exists at least one $a \in A$ such that $f(a) = b$.

**Example 0.3** Let $R$ be the set of real numbers. The function $g : R \rightarrow R$ that is specified by $g(x) = x^2$ is not surjective and is not injective. The function $h : R \rightarrow R$ that is specified by $h(x) = x^3 - x$ is surjective but not injective. The function $\exp : R \rightarrow R$ specified by $\exp(x) = e^x$ is injective, but not surjective, and the function $f : R \rightarrow R$ specified by $f(x) = x^3$ is both injective and surjective.

**Definition 0.8** Let $f : X \rightarrow Y$. There are two functions associated with $f$ that we will now define. The function $\hat{f} : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is specified by

$$\hat{f}(A) = \{ y \in Y \mid \text{for some } a \in A, y = f(a) \}.$$ 

The function $\hat{f}^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ is specified by

$$\hat{f}^{-1}(B) = \{ x \in X \mid f(x) \in B \}.$$ 

Note: Do not jump to conclusions. The function $\hat{f}^{-1}$ is not the inverse of $f$ or even the inverse of $\hat{f}$.

With the following definition we begin to pick out certain special restricted kinds of mathematical entities that were defined above. It is important that you realize that, although the idea introduced next is new relative to the above definitions, the new idea restricts the preceding definitions. Restriction buys special properties useful in important contexts, but not necessarily in all contexts. You should organize your knowledge and understanding in such a way that you automatically see these kinds of logical relationships among the ideas you know about.

**Definition 0.9** A partial ordering on set $A$ is a binary relation from $A$ to $A$ which is reflexive, anti-symmetric and transitive. (We typically use some such as symbol as $\sqsubseteq$ to denote an ordering relation, and write it in infix position. e.g. $x \sqsubseteq y.$) Specifically,
• (reflexive property) For every $a \in A$, $a \sqsubseteq a$.

• (anti-symmetric property) For every $a \in A$ and $b \in A$ if $a \sqsubseteq b$ and $b \sqsubseteq a$ then $a = b$.

• (transitive property) For every $a \in A$, $b \in A$ and $c \in A$, if $a \sqsubseteq b$ and $b \sqsubseteq c$ then $a \sqsubseteq c$.

**Definition 0.10** Let $R$ be a dyadic relation from $A$ to $B$ and $S$ a dyadic relation from $B$ to $C$. Then

$$S \circ R = \{(x, z) \in (A \times C) \mid \exists y [(x, y) \in R \text{ and } (y, z) \in S]\}$$