CSE/CIS 607 Assessment Exam 01/17/08

NAME (print): ______________________ **SIGNATURE** ______________________

Question 1: Let $f : X \longrightarrow Y$ be an injection (i.e. one-to-one). Prove that: there is a surjective (i.e. onto) function $g : Y \longrightarrow X$ such that for all $x \in X$, $g(f(x)) = x$.

Since f is one-to-one, each x will be mapped to a different y, therefore for each $y \in f(X)$, there is exactly one $x \in X$ such that $f(x) = y$. We must select a function $g : Y \rightarrow X$ such that for all $x \in X$, $g(f(x)) = x$. We can make that selection the same way we program, i.e. by giving a rule for calculating g. Here is such a rule: First, choose any one fixed element x_0 of X. Then, for each $y \in Y$, let

$$g(y) = \begin{cases} x & \text{if } y \in f(X) \text{ and } f(x) = y \\ x_0 & \text{otherwise} \end{cases}$$

For any x in X, $f(x) \in f(X)$. Thus, if we apply the above rule we obtain $g(f(x)) = x$ since $f(x) \in f(X)$ and $f(x) = f(x)$. Also, since for any x in X, $g(f(x)) = x$, g is onto.

We must show that the rule we gave actually defines a function. For this, we must show that the output given by the rule is uniquely determined by the input. The two conditions given for the rule are complementary and mutually exclusive. The first condition uniquely determines an out put since, if $y \in f(X)$, then there is exactly one x such that $f(x) = y$.

Question 2: Give an example of two sets X and Y and function f from X to Y that is one-to-one, but for which there is no surjective function g from Y to X such that $f(g(y)) = y$, for all $y \in Y$.

Let's take the set $X = \{1, 2\}$ and $Y = \{1, 2, 3, 4\}$.
And the function as $f : X \rightarrow Y$ where $f(x) = x^2$.
Therefore $f = \{(1, 1), (2, 4)\}$.
Function f is one-to-one since there is no $f(x_1) = f(x_2)$ when $x_1 \neq x_2$.
In particular, $2 \notin \mathsf{range}(f)$. Therefore, the condition $f(g(2)) = 2$ cannot be met, no matter what function g is.

Question 3: Is the following argument valid? Explain.

> Given: Clark comes from a place called Krypton or Lex retired.
> Conclusion: If Clark does not come from a place called Krypton, then the moon is made of cheese or Lex retired.

An argument is invalid when its premises(the "givens") can be true while its conclusion is false.
Our conclusion "If Clark does not come from a place called Krypton then the moon is made of cheese or Lex is retired." can only be *false* if the *if* part of the sentence is *true* and *then* part of the sentence is *false*. So we take "Clark does not come from a place called Krypton" as *true*. For the *then* part to be *false*, both 'the moon is made of cheese' and 'Lex is retired' must be *false*.
So now:

Clark comes from a place called Krypton is *false*
Lex is retired is *false*

This makes the "Given" part: *false* or *false*. Therefore the sentence is *false*. So we started to make our argument invalid by making the given *true* and conclusion *false*. However, we saw that there can't be a case like this, therefore our argument is valid.

Question 4:

Part a: Give an example of a set X for which there is no function from X to X that is one-to-one but not onto.

If we have the set $X = \{a\}$
$f : X \to X$ can be only one function, namely $f = \{(a, a)\}$
There is no other way of mapping for this set so all mappings are one-to-one and onto.

Part b: Give an example of a set Y and a function $f : Y \longrightarrow Y$ such that f is one-to-one but not onto.

Let $\mathbf{N}$ be the set of natural numbers. That is, $\mathbf{N} = \{0, 1, 2, 3, \dots\}$.
Let's define $f : \mathbf{N} \times \mathbf{N}$ where $f(x) = x^2$
The function is definitely one-to-one since this function maps each distinct element of the set of natural numbers to a distinct element in the natural numbers:
$0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 9...$
However, the set is not onto since there are natural numbers which are not squares

of other natural numbers e.g 3,5,7,124...

Question 5: Let A be the set of all bit strings of length 9 that either start with 11 or end with 0. For example, 110101001, 100110010, 111001010 are bit strings in the set A. What is the size of set A; i.e. how many members does A have?

First let's look at bit strings of length 9 that start with 11. We choose 1 for the first position, 1 for the second position. For the third position we can either have 1 or a 0; therefore 2 choices. For the fourth position again we can either have 1 or a 0; therefore 2 choices and same for the fifth position and so on.
Therefore we have $1 * 1 * 2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7$ many bit-strings.
Secondly let's look at bit strings of length 9 that end with 0. For the first position we can either have 1 or a 0; therefore 2 choices. For the second position again we can either have 1 or a 0; therefore 2 choices and same for the third position and so on till the 9^{th} position. For the 9^{th} position you have one choice: 0.
Therefore we have $2 * 2 * 2 * 2 * 2 * 2 * 2 * 2 * 1 = 2^8$ many bit-strings.
However, there are bit strings of length 9 that start with 11 and end with 0. We counted them twice since they are included in both the first and the second calculation. There are $1 * 1 * 2 * 2 * 2 * 2 * 2 * 2 * 1 = 2^6$ many of these bit-strings.
Therefore we have $2^7 + 2^8 - 2^6 = 320$ of strings of length 9 that start with either 11 or end with 0.

Appendix

Definition 0.1 For any sets A and B, the set $A \times B$, called the *Cartesian Product* of A and B, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 0.1 Let $A = \{0, 1, 2\}$ and let $B = \{0, 1\}$. Then

$$A \times B = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$

Example 0.2 Let $\mathbf{N}$ be the set of non-negative integers. That is, $\mathbf{N} = \{0, 1, 2, 3, \ldots\}$. Then

$$\mathbf{N} \times \mathbf{N} = \{(m, n) \mid m \in \mathbf{N} \textbf{ and } n \in \mathbf{N}\}.$$

Definition 0.2 For any sets A and S, A is a *subset* of S if, and only if, every member of A is a member of S. We denote that A is a subset of S by $A \subseteq S$.

Definition 0.3 For any set S, the *power set* of S, denoted by $\mathbf{P}(S)$, is the set of all subsets of S. That is,

$$\mathbf{P}(S) = \{B \mid B \subseteq S\}.$$

Note: The definition of power set implies that for anything x, $x \in \mathbf{P}(S)$ if, and only if, $x \subseteq S$.

Definition 0.4 A *relation* R from set A to set B is a subset of $A \times B$.

Definition 0.5 A relation f from A to B is called a *function* from A to B if, and only if, for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Suppose f is a function from A to B. Given $a \in A$, the $b \in B$ for which $(a, b) \in f$ is denoted by $f(a)$. The expression $f : A \longrightarrow B$ means that f is a function from A to B.

Definition 0.6 A function $f : A \longrightarrow B$ is called an *injection*, if and only if the following condition for f is true: for all elements x and y of A such that $x \neq y$: $f(x) \neq f(y)$. When a function is an injection, we also say that the function is *injective*, and we also say that the function is *one-to-one*. These phrases are equivalent ways of expressing the same thing.

Note: The definition says that $f : A \longrightarrow B$ is injective if, and only if, for every two different inputs to f we must get two different outputs. Equivalently, we cannot get the same output from f from two different inputs. The condition for f to be injective can be restated in the following equivalent form: for all elements x and y of A, if $f(x) = f(y)$, then $x = y$.

Definition 0.7 A function $f : A \longrightarrow B$ is called a *surjection*, if and only if, the following condition for f is true: for each element b of B, there exists at least one $a \in A$ such that $f(a) = b$.

Example 0.3 Let $\mathbf{R}$ be the set of real numbers. The function $g : \mathbf{R} \longrightarrow \mathbf{R}$ that is specified by $g(x) = x^2$ is not surjective and is not injective. The function $h : \mathbf{R} \longrightarrow \mathbf{R}$ that is specified by $h(x) = x^3 - x$ is surjective but not injective. The function $\exp : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $\exp(x) = e^x$ is injective, but not surjective, and the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $f(x) = x^3$ is both injective and surjective.

Definition 0.8 Let $f : X \longrightarrow Y$. There are two functions associated with f that we will now define. The function $\hat{f} : \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ is specified by

$$\hat{f}(A) = \{y \in Y \mid \text{for some } a \in A, y = f(a)\}.$$

The function $\hat{f}^{-1} : \mathbf{P}(Y) \longrightarrow \mathbf{P}(X)$ is specified by

$$\hat{f}^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Note: Do not jump to conclusions. The function $\hat{f}^{-1}$ is not the *inverse* of f or even the inverse of $\hat{f}$.

With the following definition we begin to pick out certain special restricted kinds of mathematical entities that were defined above. It is important that you realize that, although the idea introduced next is new relative to the above definitions, the new idea *restricts* the preceding definitions. Restriction buys special properties useful in important contexts, but not necessarily in all contexts. You should organize your knowledge and understanding in such a way that you automatically see these kinds of logical relationships among the ideas you know about.

Definition 0.9 A partial ordering on set A is a binary relation from A to A which is reflexive, anti-symmetric and transitive. (We typically use some such as symbol as $\sqsubseteq$ to denote an ordering relation, and write it in infix position. e.g. $x \sqsubseteq y$.) Specifically,

- (reflexive property) For every $a \in A$, $a \sqsubseteq a$.
- (anti-symmetric property) For every $a \in A$ and $b \in A$ if $a \sqsubseteq b$ and $b \sqsubseteq a$ then $a = b$.
- (transitive property) For every $a \in A$, $b \in A$ and $c \in A$, if $a \sqsubseteq b$ and $b \sqsubseteq c$ then $a \sqsubseteq c$.