CSE/CIS 607 Mathematical Basis of Computing

Assessment Exam, August 29, 2007

INSTRUCTIONS. Do all of the questions. Put your answers on the test paper. If you find yourself needing extra space, your are probably not taking a good approach.

NAME (print): ______________________________

SIGNATURE ______________________________

Question 1a: Let A be a set with exactly 6 elements. How many subsets of A are there?

Question 1b: Let A be a set with exactly 6 elements and B be a set with exactly 3 elements. How many functions are there from A to B?

Question 1c: Let A be a set with exactly 5 elements and B be a set with exactly 6 elements. Recall that a function $f : A \longrightarrow B$ is injective if, and only if, for all $u \in A$ and all $v \in A$, if $u \neq v$ then $f(u) \neq f(v)$. How many injective functions are there from A to B?

Question 2: Let $\mathbf{N}$ be the set of all non-negative integers. Give an argument that shows that there are no subsets U and V of $\mathbf{N}$ such that

$$U \times V = \{(m, n) \mid m \in \mathbf{N}, \mathrm{n} \in \mathbf{N}, \mathrm{m} \neq \mathrm{n}\}.$$

Question 3: Consider: (1) For each person, if that person has a job, then the country is healthy. (2) If there is someone who has a job, then the country is healthy.

Does (2) logically follow from (1)? Does (1) logically follow from (2)? Explain.

Question 4: Refer to the appendix for the definition of the term *partial ordering*. Consider the following definition:

Definition: Let A be a nonempty set and let $\sqsubseteq$ be a partial ordering on A. An element x of A is said to be *minimal* in the ordering $\sqsubseteq$ if, and only if, for every element y of A, if $y \sqsubseteq x$, then $x = y$. An element x of A is said to be *least* in the ordering $\sqsubseteq$ if, and only if, for every element y of A, $x \sqsubseteq y$. $\blacksquare$

Suppose that a_0 is an element of A that is *minimal* in $\sqsubseteq$, and that there is no other element of A that is also minimal in $\sqsubseteq$. Must a_0 be least in $\sqsubseteq$? Explain.

Appendix

Definition 0.1 For any sets A and B, the set $A \times B$, called the *Cartesian Product* of A and B, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 0.1 Let $A = \{0, 1, 2\}$ and let $B = \{0, 1\}$. Then

$$A \times B = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$

Example 0.2 Let $\mathbf{N}$ be the set of non-negative integers. That is, $\mathbf{N} = \{0, 1, 2, 3, \dots\}$. Then

$$\mathbf{N} \times \mathbf{N} = \{(m, n) \mid m \in \mathbf{N} \textbf{ and } n \in \mathbf{N}\}.$$

Definition 0.2 For any sets A and S, A is a *subset* of S if, and only if, every member of A is a member of S. We denote that A is a subset of S by $A \subseteq S$.

Definition 0.3 For any set S, the *power set* of S, denoted by $\mathbf{P}(S)$, is the set of all subsets of S. That is,

$$\mathbf{P}(S) = \{B \mid B \subseteq S\}.$$

Note: The definition of power set implies that for anything x, $x \in \mathbf{P}(S)$ if, and only if, $x \subseteq S$.

Definition 0.4 A *relation* R from set A to set B is a subset of $A \times B$.

Definition 0.5 A relation f from A to B is called a *function* from A to B if, and only if, for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Suppose f is a function from A to B. Given $a \in A$, the $b \in B$ for which $(a, b) \in f$ is denoted by $f(a)$. The expression $f : A \longrightarrow B$ means that f is a function from A to B.

Definition 0.6 A function $f : A \longrightarrow B$ is called an *injection*, if and only if the following condition for f is true: for all elements x and y of A such that $x \neq y$: $f(x) \neq f(y)$. When a function is an injection, we also say that the function is *injective*, and we also say that the function is *one-to-one*. These phrases are equivalent ways of expressing the same thing.

Note: The definition says that $f : A \longrightarrow B$ is injective if, and only if, for every two different inputs to f we must get two different outputs. Equivalently, we cannot get the same output from f from two different inputs. The condition for f to be injective can be restated in the following equivalent form: for all elements x and y of A, if $f(x) = f(y)$, then $x = y$.

Definition 0.7 A function $f : A \longrightarrow B$ is called a *surjection*, if and only if, the following condition for f is true: for each element b of B, there exists at least one $a \in A$ such that $f(a) = b$.

Example 0.3 Let $\mathbf{R}$ be the set of real numbers. The function $g : \mathbf{R} \longrightarrow \mathbf{R}$ that is specified by $g(x) = x^2$ is not surjective and is not injective. The function $h : \mathbf{R} \longrightarrow \mathbf{R}$ that is specified by $h(x) = x^3 - x$ is surjective but not injective. The function $\exp : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $\exp(x) = e^x$ is injective, but not surjective, and the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $f(x) = x^3$ is both injective and surjective.

Definition 0.8 Let $f : X \longrightarrow Y$. There are two functions associated with f that we will now define. The function $\hat{f} : \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ is specified by

$$\hat{f}(A) = \{y \in Y \mid \text{for some } a \in A, y = f(a)\}.$$

The function $\hat{f}^{-1} : \mathbf{P}(Y) \longrightarrow \mathbf{P}(X)$ is specified by

$$\hat{f}^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Note: Do not jump to conclusions. The function $\hat{f}^{-1}$ is not the *inverse* of f or even the inverse of $\hat{f}$.

With the following definition we begin to pick out certain special restricted kinds of mathematical entities that were defined above. It is important that you realize that, although the idea introduced next is new relative to the above definitions, the new idea *restricts* the preceding definitions. Restriction buys special properties useful in important contexts, but not necessarily in all contexts. You should organize your knowledge and understanding in such a way that you automatically see these kinds of logical relationships among the ideas you know about.

Definition 0.9 A partial ordering on set A is a binary relation from A to A which is reflexive, anti-symmetric and transitive. (We typically use some such as symbol as $\sqsubseteq$ to denote an ordering relation, and write it in infix position. e.g. $x \sqsubseteq y$.) Specifically,

- (reflexive property) For every $a \in A$, $a \sqsubseteq a$.
- (anti-symmetric property) For every $a \in A$ and $b \in A$ if $a \sqsubseteq b$ and $b \sqsubseteq a$ then $a = b$.
- (transitive property) For every $a \in A$, $b \in A$ and $c \in A$, if $a \sqsubseteq b$ and $b \sqsubseteq c$ then $a \sqsubseteq c$.