CSE/CIS 607 Mathematical Basis of Computing
Practice Exam II

Question 1: Is the following argument valid? Explain.

> Given: Clark comes from a place called Krypton or Lex retired.
> Conclusion: If Clark does not come from a place called Krypton, then the moon is made of cheese or Lex retired.

Answer: Let K be "Clark comes from a place called Krypton". Let L be "Lex retired". Let C be "The moon is made of cheese". The argument has the form:

premise:	$K \vee L$
conclusion:	$\neg K \supset (C \vee L)$

By Gentzen's method, we can see that if we try to falsify the conclusion whole holding the premise true, it must be that $\neg K$ is true, but $C \vee L$ is false. Therefore, K is false and L is false. Therefore the $K \vee L$ is false, which contradicts that $K \vee L$ is true. Thus the conclusion must be true if the premise is true. Hence, the argument is valid.

Question 2: Give a formal tableau proof that shows that the following tableau is valid.

g1. $(P \supset (Q \supset R)) \supset ((P \wedge \neg R) \supset \neg Q)$ given

Answer:

	g1. $(P \supset (Q \supset R)) \supset ((P \wedge \neg R) \supset \neg Q)$	given
a1. $P \supset (Q \supset R)$		g1 if-split
	g2. $(P \wedge \neg R) \supset \neg Q$	g1 if-split
a2. $P \wedge \neg R$		g2 if-split
	g3. $\neg Q$	g2 if-split
a3. P		a3 $\wedge$-split
a4. $\neg R$		a3 $\wedge$-split
a5. $\mathsf{False} \vee (\mathsf{True} \supset (Q \supset R))$		a1, a3 AA
a6. $Q \supset R$		a5 rew
a7. $(Q \supset \mathsf{False}) \vee \neg\mathsf{True}$		a6, a4 AA
a8. $\neg Q$		a7 rew
	g4. $\neg\mathsf{False} \wedge \mathsf{True}$	a8, g3 AG
	g5. True	g4 rew

Question 3: What's wrong with the following proposed inference rule? Let P be a common subformula of assertion $\mathcal{A}$ and goal $\mathcal{G}$, occurring at least once in each of these formulas. Then

From goal $\mathcal{G}_1$ and goal $\mathcal{G}_2$
derive goal $\mathcal{G}_1[P \mapsto False] \vee \mathcal{G}_2[P \mapsto True]$

(This is similar to, but not the same as, GG-resolution.)

Answer: The following solution is due fundamentally to Robert C. Grant, and has the virtue of conforming to the polarity strategy.

g1. $\neg P$	given
g2. $P \wedge \mathsf{False}$	g2 if-split

This tableau is not valid since the tableau's corresponding formula is not valid:

$$\neg P \vee (P \wedge \mathsf{False})$$

is not valid. The formula is not valid since it can be falsified by assigning T to P. However, if we apply the proposed inference rule to the tableau, we obtain

g1. $\neg P$	given
g2. $P \wedge \mathsf{False}$	g2 if-split
g3. $\neg\mathsf{False} \vee (\mathsf{True} \wedge \mathsf{False})$	g1, g2
g4. True	g3 rew

The resulting tableau is valid. A rule which produces a valid tableau from a given invalid one cannot correctly tell us that the given tableau is valid, and is therefore unsound. For a rule of inference to be sound, it must that for any two tableau $\mathcal{T}_1$ and $\mathcal{T}_2$, **if** $\mathcal{T}_2$ results from $\mathcal{T}_1$ by applying the rule **and if** $\mathcal{T}_2$ is valid, **then** $\mathcal{T}_1$ is also valid. This tells us that if we produce a valid tableau from a given tableau by applying sound rules, then the tableau we were given must be valid.

Question 4: Intentionally omitted.

Question 5: Consider:

> **Assertion 1:** If Bob teleports his qubit to Alice, and Alice receives Bob's qubit or Alice is spied on, then Alice will send a query to Bob.
> **Assertion 2:** If Alice teleports her qubit to Bob, then Bob teleports his qubit to Alice and Alice receives Bob's qubit.
> **Assertion 3:** Alice will not send a query to Bob.
> **Goal:** Therefore, if Alice teleports her qubit to Bob, then Alice is spied on.

Let B stand for "Bob teleports his qubit to Alice."
Let R stand for "Alice receives Bob's qubit."
Let S stand for "Alice is spied on."
Let Q stand for "Alice will send a query to Bob."
Let A stand for "Alice teleports her qubit to Bob."

(1) Express the assertions and goal in our notation for propositional logic.

(2) Does the goal follow from the assertion? (Explain.)

Question 6: Is the following tableau valid? Explain. (You **do not** need to give a tableau proof if is valid.)

a1. $(J \wedge K) \supset L$	given
a2. $(J \supset L) \supset M$	given
a3. $\neg K \vee N$	given
g1. $K \supset (M \wedge N)$	given

Answer: The tableau is valid. The tableau would not be valid if all of its goals can be falsified while the assertions are true. Suppose $K \supset (M \wedge N)$ is false and the assertions are true. Then K is true and $M \wedge N$ is false. Since $\neg K \vee N$ is true, but $\neg K$ is false, N is true. Since, $M \wedge N$ is false and N is true, M is false. Since $(J \supset L) \supset M$ is true, but M is false, $J \supset L$ is false. Since $J \supset L$ is false, J is true and L is false. Since, $(J \wedge K) \supset L$ is true, but L is false, $J \wedge K$ is false. Since $J \wedge K$ is false, and J is true, K is false. But K is true and we have a contradiction. Therefore if we suppose the tableau is not valid, a contradiction results.

Question 7: Give a formal tableau proof that shows that the following tableau is valid.

a1. $(A \vee B) \supset (C \wedge D)$	given
a2. $\neg C$	given
g1. $\neg B$	given

Answer:

a1. $(A \vee B) \supset (C \wedge D)$	given
a2. $\neg C$	given
g1. $\neg B$	given
a3. $((A \vee B) \supset (\mathsf{False} \wedge D)) \vee \neg\mathsf{True}$	a1, a2 AA
a4. $\neg A \wedge \neg B$	a3 rew
a5. $\neg B$	a4 $\wedge$-split
g2. $\neg\mathsf{False} \wedge \mathsf{True}$	a5, g2 AG
g3. True	g2 rew

Question 8: Give a formal tableau proof that shows that the following tableau is

valid.

a1.	$Q \supset \neg P$	given
a2.	$P \supset Q$	given
g1.	$\neg P$	given

Answer:

a1.	$Q \supset \neg P$	given
a2.	$P \supset Q$	given
g1.	$\neg P$	given
g2.	$\neg(\mathsf{True} \supset Q) \wedge \neg\mathsf{False}$	g1, a2 GA
g3.	$\neg Q$	g2 rew
g4.	$\neg(\mathsf{True} \supset \neg P) \wedge \neg\mathsf{False}$	g3, a1 GA
g5.	P	g4 rew
g6.	$\neg\mathsf{False} \wedge \mathsf{True}$	g1, g6 GG
g7.	True	g6 rew

Question 9: What's wrong with the following proposed inference rule? Let P be a common subformula of assertion $\mathcal{A}_1$ and assertion $\mathcal{A}_2$. Then

> From assertion $\mathcal{A}_1$ and assertion $\mathcal{A}_2$
> derive assertion $\mathcal{A}_1[P \mapsto False] \wedge \mathcal{A}_2[P \mapsto True]$

Question 10: Consider:

> **Assertion 1:** If Alice doesn't catch the bus, then Bob gambles away his money or Bob leaves town.
>
> **Assertion 2:** If Alice catches the bus and Bob does not leave town, then Bob gambles away his money.
>
> **Assertion 3:** If Bob gambles away his money, then Bob leaves town.
>
> **Goal:** Bob leaves town.

Let A stand for "Alice catches the bus." Let B stand for "Bob gambles away his money." Let L stand for "Bob leaves town."

(1) Express the assertions and goal in our notation for propositional logic.

(2) Does the goal follow from the assertion? (Explain.)

Question 11: Is the following tableau valid? If it is, give a formal tableau proof. If it is not, explain why. (Similar to the formula in Question 2, except the rightmost occurrence of P in Question 2 is replaced by Q.

g1. $(P \supset (Q \supset \neg R)) \supset ((P \supset Q) \supset (R \supset \neg Q))$

Answer: The tableau is not valid because all of the goals can be falsified while all of the assertions while all of the assertions can be true: Assign F to P and assign T to R and Q. Then (g1) is false.

Question 12: Give a formal tableau proof that shows that the following tableau is valid.

a1. $(P \wedge Q) \supset R$	given
a2. P	given
a3. $\neg Q \supset (\neg P \vee R)$	given
g1. R	given

Answer:

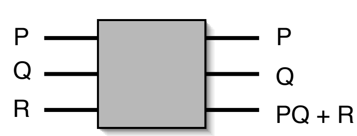
a1. $(P \wedge Q) \supset R$	given
a2. P	given
a3. $\neg Q \supset (\neg P \vee R)$	given
g1. R	given
a4. $\neg P \vee \neg Q \vee R$	a1 rew
a5. $\neg P \vee Q \vee R$	a3 rew
a6. $\mathsf{False} \vee (\neg \mathsf{True} \vee \neg Q \vee R$	a2, a4 AA
a7. $\neg Q \vee R$	a6 rew
a8. $\mathsf{False} \vee (\neg \mathsf{True} \vee Q \vee R$	a2, a5 AA
a9. $Q \vee R$	a2, a5 AA
a10. $\mathsf{False} \vee (\neg \mathsf{True} \vee R)$	a9, a7 AA
a11. R	a10 rew
g2. $\neg \mathsf{False} \wedge \mathsf{True}$	a11, g2 AG
g3. True	g2 rew

Question 13: What's wrong with the following proposed inference rule? Let P be a common subformula of assertion $\mathcal{A}$ and goal $\mathcal{G}$, occurring at least once in each of these formulas. Then

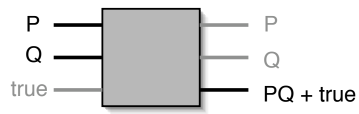
From assertion $\mathcal{A}$ and goal $\mathcal{G}$
derive goal $\mathcal{A}[P \mapsto False] \wedge \mathcal{G}[P \mapsto True]$

(This is similar to, but not the same as, AG-resolution.)

Question 14: This question is optional - there will not be a question like it on the upcoming exam. BUT, making an effort to do the question will help you reinforce your understanding of propositional logic. The diagram below denotes a *Toffoli gate*. P, Q and R are Boolean-valued. The expression $\mathsf{PQ} + \mathsf{R}$ in the diagram denotes the Boolean-value of the propositional formula $(\mathsf{P} \wedge \mathsf{Q}) \oplus \mathsf{R}$. ($\oplus$ denotes exclusive-or.)



Notice that if the value of R is fixed at true, then the Toffoli gate acts as a NAND gate:



Show how to make the Toffoli gate act as an XOR gate. XOR(A,B) is equivalent to the propositional formula

$$A \oplus B$$

Question 15: Consider:

> **Assertion 1:** If Alice sends an encrypted text to Bob, then Bob will be safe and Bob doesn't leave town.
>
> **Assertion 2:** If Alice does not send an encrypted text to Bob and Bob leaves town, then Bob will not be safe.
>
> **Assertion 3:** Bob doesn't leave town or Bob is safe.
>
> **Goal:** Bob does not leave town.

Let A stand for "Alice sends an encrypted text to Bob." Let B stand for "Bob will be safe." Let L stand for "Bob leaves town."

(1) Express the assertions and goal in our notation for propositional logic.

(2) Does the goal follow from the assertion? (Explain.)

Question 16: Is the following tableau valid? If it is, give a formal tableau proof. If it is not, explain why.

g1. $(P \supset (Q \supset \neg R)) \supset ((Q \supset \neg P) \supset (P \supset R))$

Answer: The only goal in the tableau can be falsified, while all the assertions are true: Assign F to Q and R and T to P.

Question 16a: Give a formal tableau proof that shows that the following tableau is valid.

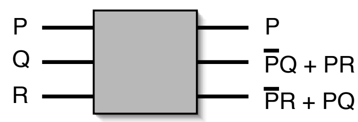
a1. $P \supset (\neg Q \vee R)$	given
a2. $\neg R$	given
a3. $P \supset (Q \vee R)$	given
g1. $\neg P$	given

Answer:

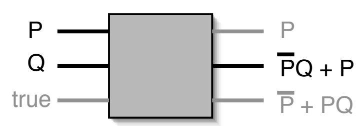
a1. $P \supset (\neg Q \vee R)$	given
a2. $\neg R$	given
a3. $P \supset (Q \vee R)$	given
g1. $\neg P$	given
g2. $\neg(\mathsf{True} \supset (Q \vee R)) \wedge \neg\mathsf{False}$	g1, a3 GA
g3. $\neg(Q \vee R)$	g2, rew
g4. $\neg(\mathsf{True} \supset (\neg Q \vee R)) \wedge \neg\mathsf{False}$	g1, a1 GA
g5. $\neg(\neg Q \vee R)$	g2, rew
g6. $\neg(\mathsf{False} \vee R) \vee \neg(\neg\mathsf{True} \vee R)$	g3, g5 GG
g7. $\neg R$	g5, rew
g8. $\neg\mathsf{False} \wedge \mathsf{True}$	a2, g7 AG
g9. True	g8 rew

Question 17: Intentionally omitted.

Question 18: This question is optional - there will not be a question like it on the upcoming exam. BUT, making an effort to do the question will help you reinforce your understanding of propositional logic. The diagram below denotes a *Fredkin gate*. P, Q and R are Boolean-valued. The expression $\overline{\mathsf{P}}\mathsf{Q} + \mathsf{PR}$ in the diagram denotes the Boolean-value of the propositional formula $(\neg \mathsf{P} \wedge \mathsf{Q}) \oplus (\mathsf{P} \wedge \mathsf{R})$. ($\oplus$ denotes exclusive-or.)



Notice that if the value of R is fixed at true, then the Fredkin gate acts as an OR gate:



Show how to make the Fredkin gate act as an AND gate.

Question 19: Use the unification algorithm to determine whether there is a most

general unifying substitution that solves the equation

$$k(f(x,u,a), f(g(y),y,v)) = k(f(g(y),y,v), f(w,h(z),z))$$

In the preceding equation, a is a constant, and x, y, z, u, v and w are variables.

Answers: $k(f(x,u,a), f(g(y),y,v)) = k(f(g(y),y,v), f(w,h(z),z))$

$f(x,u,a) = f(g(y),y,v)$
$f(g(y),y,v) = f(w,h(z),z)$

$x = g(y)$

$u = y$
$a = v$
$g(y) = w$
$y = h(z)$
$v = z$

$x = g(h(z))$
$u = h(z)$
$v = a$
$w = g(h(z))$
$y = h(z)$
$z = a$

$x = g(h(a))$
$u = h(a)$
$v = a$
$w = g(h(a))$
$y = h(a)$
$z = a$

$k(f(g(h(a)), h(a), a), f(g(h(a)), h(a), a)) = k(f(g(h(a)), h(a), a), f(g(h(a)), h(a), a))$

Question 20: Use the unification algorithm to determine whether there is a most general unifying substitution that solves the equation

$$j(g(x, u, w), g(f(y), y, v)) = j(g(f(y), y, v), g(x, k(z), z))$$

In the preceding equation, x, y, z, u, v and w are variables.

Answer:
$j(g(x, u, w), g(f(y), y, v)) = j(g(f(y), y, v), g(x, k(z), z))$

$g(x, u, w) = g(f(y), y, v)$
$g(f(y), y, v) = g(x, k(z), z)$

$x = f(y)$
$u = y$
$w = v$
$f(y) = x$
$y = k(z)$
$v = z$

$x = f(k(z))$
$u = k(z)$
$w = z$
$y = k(z)$
$v = z$

$j(g(f(k(z)), k(z), z), g(f(k(z)), k(z), z)) = j(g(f(k(z)), k(z), z), g(f(k(z)), k(z), z))$

Question 21: Assign polarity to each of the subformula occurrences in the following formula

$$(\neg P \supset ((Q \equiv P) \supset Q)) \vee (P \wedge \neg R)$$

Answer:

$$(\neg^- P^+ \supset^+ ((Q^\pm \equiv^- P^\pm) \supset^+ Q^+)) \vee^+ (P^+ \wedge^+ \neg^+ R^-)$$

Question 22: Let Ψ be a formula in which the subformula $P \vee \neg P$ has exactly one occurrence, and that occurrence has negative polarity in Ψ. Is the following formula valid? If it is valid, explain why. If it is not valid give a (simple) counterexample.

$$(\Psi[P \vee \neg P] \wedge Q) \supset (\Psi[P \wedge \neg P] \wedge Q)$$

Question 23: Find a most general unifier of the two terms

$$g(g(w, g(b, z)), w) \text{ and } g(g(y, g(b, u)), g(u, z)) .$$

Answer:

$g(g(w, g(b, z)), w) = g(g(y, g(b, u)), g(u, z))$

$g(w, g(b, z)) = g(y, g(b, u))$
$w = g(u, z)$

$w = y$
$g(b, z) = g(b, u)$
$w = g(u, z)$

$w = y$
$g(b, z) = g(b, u)$
$y = g(u, z)$

$w = y$
$b = b$
$z = u$
$y = g(u, z)$

$w = y$
$z = u$
$y = g(u, z)$

$w = y$
$z = u$

$y = g(u, u)$

$w = g(u, u)$
$z = u$
$y = g(u, u)$

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g(g(g(u,u),g(b,u)),g(u,u)) = g(g(g(u,u),g(b,u)),g(u,u))

Question 24: Find a most general unifier of the two terms

$$g(x, y, f(x), z, u) \quad \text{and} \quad g(a, x, z, f(x), h(v, z))\,.$$

Answer:
$g(x, y, f(x), z, u) = g(a, x, z, f(x), h(v, z))$

$x = a$
$y = x$
$f(x) = z$
$z = f(x)$
$u = h(v, z)$

$x = a$
$y = a$
$f(a) = z$
$z = f(a)$
$u = h(v, z)$

$x = a$
$y = a$
$z = f(a)$
$z = f(a)$
$u = h(v, z)$

$x = a$
$y = a$

$z = f(a)$
$f(a) = f(a)$
$u = h(v, f(a))$

$x = a$
$y = a$
$z = f(a)$
$a = a$
$u = h(v, f(a))$

$x = a$
$y = a$
$z = f(a)$
$u = h(v, f(a))$

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check: $g(a, a, f(a), f(a), h(v, f(a))) = g(a, a, f(a), f(a), h(v, f(a)))$

Question 25: Assign polarity to each of the subformula occurrences in the following formula.

$$(\neg P \supset (Q \supset (Q \equiv P))) \wedge \neg(P \supset \neg R)$$

Answer:

$$(\neg^- P^+ \supset^+ (Q^- \supset^+ (Q^\pm \equiv^+ P^\pm))) \wedge^+ \neg^+(P^+ \supset^- \neg^- R^+)$$

Question 26: Find a most general unifier of the two terms

$$f(h(y, x, g(v, x))), \qquad f(h(v, x, g(g(u, u), f(u)))) \,.$$

Answer:
$f(h(y, x, g(v, x))) = f(h(v, x, g(g(u, u), f(u))))$

$h(y, x, g(v, x)) = h(v, x, g(g(u, u), f(u)))$

$y = v$
$x = x$
$g(v, x) = g(g(u, u), f(u))$

$y = v$
$g(v, x) = g(g(u, u), f(u))$

$y = v$
$v = g(u, u)$
$x = f(u)$

$y = g(u, u)$
$v = g(u, u)$
$x = f(u)$

$f(h(g(u, u), f(u), g(g(u, u), f(u)))) = f(h(g(u, u), f(u), g(g(u, u), f(u))))$

Question 27: Intentionally omitted

Question 28: Is the following tableau valid? If it is, give a formal tableau proof. If it is not, explain why.
g1. $(P \supset (Q \wedge R)) \supset ((P \supset Q) \supset (P \supset R))$.

Answer: The tableau is valid.

g1. $(P \supset (Q \wedge R)) \supset ((P \supset Q) \supset (P \supset R))$	given
g2. True	g1 rew

(Seriously!)

Question 29: Is the following tableau valid? If it is, give a formal tableau proof. If it is not, explain why.

g1. $((P \supset Q) \supset (P \supset R)) \supset (P \supset (Q \wedge R))$.

Answer: The tableau is not valid because the goal can be falsified while all of the assertions are true: Assign T to P and R and F to Q.

Question 30: Give a formal tableau proof that shows that the following tableau is valid.

a1. $P \supset (Q \equiv P)$	given
a2. $\neg P \supset R$	given
g1. $R \vee (Q \equiv P)$	given

(**Hint:** You do **not** need to use the equivalence rule, and you do **not** need to eliminate the equivalances with rewrites.)

Answer:

a1. $P \supset (Q \equiv P)$	given
a2. $\neg P \supset R$	given
g1. $R \vee (Q \equiv P)$	given
a3. $(\neg \mathsf{False} \supset R) \vee (\mathsf{True} \supset (Q \equiv P))$	a2, a1 AA
a4. $R \vee (Q \equiv P)$	a3 rew
g2. $\neg \mathsf{False} \wedge \mathsf{True}$	a4, g1 AG
g3. True	g2 rew

Question 31: Intentionally omitted.

Question 32: Consider:

> **Assertion:** The bus is late.
> **Goal:** If the bus is not late, then Bill is a dolphin.

Let L stand for "The bus is late." Let D stand for "Bill is a dolphin." **(1)** Express the assertion and goal in our notation for propositional logic. **(2)** Does the goal follow from the assertion? (Explain.)

Answer: The goal is equivalent to

The bus is late or Bill is a dolphin

which follows from the assertion

The bus is late

Question 33: Is the following tableau valid? If it is, give a formal tableau proof. If it is not, explain why.
a1. $P \supset (Q \supset R)$
a2. $(R \wedge S) \supset \neg U$
a3. $\neg V \supset (S \wedge U)$
g1. $(R \wedge U) \supset V$

Answer: The tableau is valid: Suppose (g1) is false. Then R and U are true, but V is false. Therefore, $\neg V$ is true. By (a3), S and U are true. Therefore, $\neg U$ is false. Thus (a2) is false. Hence, if the goal is false, at least one of the assertions must be false.

Question 34: Is the following tableau valid? If it is, give a formal tableau proof. If it is not, explain why.

$$
\begin{array}{rl}
\text{a1.} & P \supset (Q \supset R) \\
\text{a2.} & (R \wedge S) \supset \neg U \\
\text{a3.} & \neg V \supset (S \wedge U) \\
\text{g1.} & Q \supset (\neg P \vee \neg R)
\end{array}
$$

Answer: The tableau is not valid. Let T be assigned to P, Q, R, S, and V, and let F be assigned to U. Then the goal is false, and the assertions are true.

Question 35: Suppose that subformula P occurs at least once in each of the formulas $\mathcal{A}$ and $\mathcal{B}$. Explain why the following tableau is valid:

$$\text{g1. } (\mathcal{A} \wedge \mathcal{B}) \supset (\mathcal{A}[P \mapsto \mathsf{True}] \vee \mathcal{B}[P \mapsto \mathsf{False}])$$

Answer:

	g1. $(\mathcal{A} \wedge \mathcal{B}) \supset (\mathcal{A}[P \mapsto True] \vee \mathcal{B}[P \mapsto False])$	given
a1. $\mathcal{A} \wedge \mathcal{B}$		g1 if-split
	g2. $\mathcal{A}[P \mapsto \mathsf{True}] \vee \mathcal{B}[P \mapsto \mathsf{False}]$	g1 if-split
a2. $\mathcal{A}$		a1 $\wedge$-split
a3. $\mathcal{B}$		a1 $\wedge$-split
a4. $\mathcal{A}[P \mapsto \mathsf{True}] \vee \mathcal{B}[P \mapsto \mathsf{False}]$		a2, a3 AA
	g3. $\neg\mathsf{False} \wedge \mathsf{True}$	a4, g2 AG
	g4. True	g3 rew