Practice Exam 1

CIS/CSE 607, Spring 2009

Problem 1) Let R be a reflexive binary relation on a set A. Prove that R is transitive if, and only if, $R = R \circ R$.

Problem 2) Give an example of a transitive binary relation on a set A for which $R \neq R \circ R$

Problem 3) How many binary relations are there, on a set with 3 elements, that have all three properties: reflexive, antisymmetric and transitive? In other words, how many partial orderings are there on a set with 3 elements?

Problem 4) Consider the set of all permutations of a set A. [Remember that a permutation of A is just a one-to-one and onto function from A to A.] Explain why a permutation of A is a binary relation on A. Now let f be the permutation of the set $\{1, 2, 3\}$ defined by the input-output pairs $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$. Give the input-output pairs for f^2, f^3 and f^{-1}.

Problem 5) Consider again the set of all permutations of a set A. Let I be the identity permutation: i.e. $I(x) = x$ for all $x \in A$. Let π_0 be one of these permutations. Consider the set of permutations

$$\{\ldots, (\pi_0^{-1})^2, \pi_0^{-1}, I, \pi_0, \pi_0^2, \ldots, \pi_0^n, \ldots\}$$

For two permutations f and g (possibly, $f = g$) of A, let $f \sim g$ if, and only if, there is an integer k such that $f \circ g^{-1} = \pi_0^k$. Prove that $\sim$ is an equivalence relation on the set of all permutations of A. [For permutation f of A, f^{-1}

is the inverse permutation. That is, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$, for all $x \in A$.]

Problem 6) Let $A = \{1, 2, 3\}$. List all six permutations of A. List the equivalence classes of the $\sim$ relation defined in problem 5, where π_0 is the permutation f given in problem 4.

Problem 7) Recall from exam 1: Let U be a nonempty set. Let $\blacktriangleright$ be the operation defined on the power set of U as follows: for each subset A of U and each subset B of U, let

$$A \blacktriangleright B = (U - A) \cup B$$

Let U be a nonempty set (i.e. $U \neq \emptyset$). For any function f with domain U and codomain $\{0, 1\}$ the preimage $f^{-1}(\{1\})$ of the element 1 in the codomain is a subset of U. Conversely, given any subset A of U, the *characteristic function* of A is defined by

$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A \end{cases}$$

These remarks establish a one-to-one correspondence between the power set of U and the set of functions with domain U and codomain $\{0, 1\}$. For subsets A and B, the following formulas express the characteristic function of their difference, union, and intersection in terms of the characteristic functions χ_A and χ_B. ($*$ is multiplication.)

$$\chi_{U-A}(x) = 1 - \chi_A(x)$$

$$\chi_{A \cap B}(x) = \chi_A(x) * \chi_B(x)$$

$$\chi_{A \oplus B}(x) = \chi_A(x) + \chi_B(x) - 2 * \chi_A(x) * \chi_B(x)$$

$$\chi_{A-B}(x) = \chi_A(x) * (1 - \chi_B(x))$$

Give a similar formula for $\chi_{A \blacktriangleright B}$ in terms of χ_A and χ_B.

Problem 8) Let

$$A = \{\emptyset, \{\emptyset\}\}$$

Answer T (true) of F (false) legibly for each line, below. The first one is done for you.

There is one element in A. **false**

$$\{\emptyset\} \in A$$

$$\{\emptyset\} \subseteq A$$

$$\{\emptyset, \{\emptyset\}\} = \{\emptyset\}$$

$$\emptyset \in A$$

$$\emptyset \subseteq A$$

$$\{\{\emptyset\}\} = \{\emptyset\}$$

$$\emptyset \cup \{\emptyset\} = \emptyset$$

Problem 9) Suppose that δ_{ij}, $i = 1, \ldots, 100$, $j = 1, \ldots, 100$ is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Evaluate

$$\sum_{i=1}^{100} \sum_{j=1}^{100} \frac{1}{2} \delta_{ij} (i + j)$$

Problem 10) Prove by induction: For every natural number $n > 6$, $3^n < n!$.

Problem 11) Recall the Fibonacci numbers: $f_0 = 0$, $f_1 = 1$, $f_n = f_{n-2} + f_{n-1}$, where $n \geq 2$. For which values of n is $f_n \textbf{ mod } 3 = 0$? Prove your answer by induction.

Problem 12) How many bit strings of length 17 are there that contain exactly 5 0's and exactly 12 1's?

Problem 13) Let A and B be finite sets. Assume that the number of elements in A is m and the number of elements in B is n. Also assume $m \geq n$. Let $g : B \longrightarrow A$ be a one-to-one function. In terms of m and n, how many functions $f : A \longrightarrow B$ are there such that the following condition is satisfied: $f(g(x)) = x$ for every x in B? *Briefly* explain your reasoning.

Problem 14) Let $\mathbf{Z}$ be the set of integers, and let

$$P = \{n \in \mathbf{Z} \mid 1 \leq n \leq 20 \text{ and } n \text{ is odd}\}$$

and

$$Q = \{n \in \mathbf{Z} \mid 1 \leq n \leq 20 \text{ and } n \text{ is a multiple of } 5\}$$

For any two sets U and V, let

$$U - V = \{u \in U \mid u \notin V\}$$

and

$$U \Delta V = (U \bigcup V) - (U \bigcap V)\,.$$

List the elements in $P \Delta Q$.

Problem 15) Recall that the product $P \times Q$ of two sets P and Q is a set of ordered pairs determined by the following definition:

$$P \times Q = \{(p, q) \mid p \in P \text{ and } q \in Q\}$$

Let S, A and B be sets such that $S \subseteq A \times B$. Suppose $S = \{(0,0),(1,1)\}$. Prove that there are no sets D and E such that $D \subseteq A$, $E \subseteq B$, and $S = D \times E$.

Problem 16) A (binary) relation R over a set A is simply a set of pairs of elements drawn from A; that is, $R \subseteq A \times A$.

The *inverse* of a relation R is the relation R^{-1} defined by:

$$R^{-1} = \{(x,y) \mid (y,x) \in R\}$$

The *composition* $R_1 \circ R_2$ of relations R_1 and R_2 is defined by:

$$R_1 \circ R_2 = \{(x,z) \mid \exists y. \ (\ (x,y) \in R_1 \wedge (y,z) \in R_2)\}.$$

R^n is the result of composing n copies of R. More precisely, R^n is defined as follows:

$$\begin{aligned} R^1 &= R \\ R^{n+1} &= R \circ R^n \qquad \text{(for } n \geq 1) \end{aligned}$$

A relation R is *transitive* if it satisfies the following property:

$$\forall x,y,z. \ [(x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R]$$

Let T and U be relations over the set $A = \{1,2,3,4\}$, as follows:

$$\begin{aligned} T &= \{(1,1),(2,1),(3,3),(4,4),(3,4)\} \\ U &= \{(2,4),(1,3),(3,3),(3,2)\} \end{aligned}$$

Calculate the following relations:

1. T^{-1}
2. $U \circ T$
3. U^3

Problem 17) Is the following statment true?

The composition of two transitive relations is always transitive.

Support your answer by giving either a rigorous proof of its validity (if you answer "yes") or a convincing/explained counterexample (if you answer "no").

Problem 18) Consider the following definitions: (a) Let S be the set $\{1, 2, 3, 4\}$. How many binary relations on S are there?

(b) An ardering on a set is a transitive binary relation on the set. Give an example of an ordering on set S that is antisymmetric but not refelxive.

(c) Give an example of an ordering on set S that is reflexive but not antisymmetric.

(d) Give an example of a reflexive, antisymmetric binary relation on S that is not an ordering.

Problem 19) Let $S = \{1, 2, 3, 4\}$.

(a) How many functions are there from S to S?

(b) How many bijections are there from S to S?

Problem 20)

Definition: (directed set). Let $A \subseteq D$ where D is a partial order. A is *directed* in D iff A is nonempty and: for all $x \in A$ and for all $y \in A$ there exists $z \in A$ such that $x \sqsubseteq z$ and $y \sqsubseteq z$.

Definition: (dcpo). A partial order D is a *dcpo* (i.e. a *directed-complete partial order*) iff every directed subset of D has a least upper bound. We denote the least upper bound of a set A (directed or not) by $\sqcup A$.

Definition: (way below, the order of approximation). Let D be a dcpo. The ordering $\ll$ is defined by: $x \ll y$ iff for all directed subsets A in D, if $y \sqsubseteq \sqcup A$ then for some $z \in A$, $x \sqsubseteq z$.

The ordering $\ll$ is called the *order of approximation*. The expression $x \ll y$

is read as x is *way below* y.

The set of all elements $\{y \mid y \ll x\}$ is denoted by $\downarrow\!\!x$.

Definition: (compact element) Let D be a dcpo. An element x of D is *compact* iff $x \ll x$.

Definition: (continuous domain). A *continuous* dcpo (also called a *continuous domain*) is a dcpo such that for every $x \in D$, $x = \sqcup(\downarrow\!\!x)$

Definition: The *successor* of a set S is the set $S \cup \{S\}$.

(a) Let ω be the set of nonnegative integers. Consider the successor of ω, which we denote by ω', with the ordering obtained by extending the natural ordering on the nonnegative integers so that $x \leq \omega$ for all elements x in ω'. Show that the resulting system is a continuous dcpo.

(b) Are there any compact elements in ω'? Which element(s) is (are) compact? Explain.

One way to get an insight into answering these questions is to determine for yourself what the directed subsets of ω' look like.

Problem 21) A *trit* is an element of the set $\{0, 1, 2\}$. Let us call a string σ of trits a *good* trit-string iff σ does not have two consecutive $0's$ in it anywhere. How many good trit-strings are there of length 6.