The Mathematical Basis of Computing:
Final Problem Set
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Due: Monday, May 17, 2010, 5pm

Problem 1: Skolemize as an assertion.
\[ \forall x \forall y \forall z \exists w p(f(y), g(a, z, w), x, w) \]

Problem 2: Let \( T \) be a theory with the language without constant and function symbols and whose only predicate symbols are \( p \) and \( = \). Suppose that the only nonlogical axiom of \( T \) is
\[ \forall x \exists y p(x, y) \]
Explain why
\[ \exists y \forall x p(y, x) \]
is not a theorem of \( T \).

Problem 3: Consider the following argument:

Premise 1: Some hobbit who is a hermit is feared by all sheep.
Premise 2: Baggins is a hobbit.
Premise 3: Lambchop is a sheep.
Therefore, if Lambchop does not fear Baggins, then Baggins is not a hermit.

Is the argument valid? Explain.

Legend: Let \( p(x) \) stand for \( x \) is a hobbit; \( r(x) \) stand for \( x \) is a hermit; \( s(x) \) stand for \( x \) is a sheep; \( b \) stand for Baggins; \( c \) stand for Lambchop; and \( q(x, y) \) stand for \( x \) fears \( y \).
Problem 4: Apply the unification algorithm to produce a most-general unifier which solves the equation given below, if a solution exists, and indicate failure otherwise.

\[ g(x, f(y, a), y) = g(f(f(a, z), z), x, f(z, a)) \]

Problem 5: We have seen that Robinson’s theory of arithmetic Q is too weak to prove familiar properties of the natural numbers such as

\[ \forall x \forall y [x + y = y + x] \]

If we add induction axioms to Q we can overcome some, but not all, limitations Q. There is an induction axiom for each formula of the form

\[ \forall v A(v) \]

in the language of arithmetic, where the variable \( v \) in the outer-most universal quantifier is the only variable that occurs free in \( A(x) \). For example, if \( \forall y B(y) \) is

\[ \forall y \exists w \forall x [\exists u (x * u = y + s(w))] \rightarrow (x = s(0) \lor (x = y + s(w))) \]

then \( B(y) \) itself is

\[ \exists w \forall x [\exists u (x * u = y + s(w))] \rightarrow (x = s(0) \lor (x = y + s(w))) \]

The induction axiom corresponding to \( A(x) \) is

\[ (A(0) \land \forall x [A(x) \rightarrow A(s(x))]) \rightarrow \forall x, A(x) \]

Write out the induction axiom for \( B(y) \).

Problem 6: Expand the following formula over a universe with the two individuals \{0, 1\}.

\[ p(a) \land \forall x [p(f(f(x))) \land p(x)] \rightarrow p(f(x)) \land \forall x [\neg p(f(x)) \rightarrow p(x)] \land \neg p(f(a)) \]

Can this formula be satisfied over a universe with two individuals; i.e. can \( a, p \) and \( f \) be interpreted so that the formula evaluates to true in such a universe.
**Problem 7:** Let $A$ be a set and let $(R, A, A)$ be a dyadic (yet another word for a binary, or 2-place) relation on $A$; i.e. $R \subseteq (A \times A)$. Let $p$ and $q$ be (not necessarily distinct) elements of $A$. The interval $[p, q]$ of $(R, A, A)$ is defined by

$$[p, q] = \{ a \in A \mid (p, a) \in R \text{ and } (a, q) \in R \}$$

Let $x \in A$ and let $[p_1, q_1]$ and $[p_2, q_2]$ be intervals of $(R, A, A)$ that contain $x$. Must there be an interval $[a, b]$ of $(R, A, A)$ that contains $x$ and is a subset of $[p_1, q_1] \cap [p_2, q_2]$?

If there must be such an interval, prove it. If not, give a counterexample.

**Problem 8:** A causal set is a partially ordered set (look up the definition of partially ordered set if you don’t recall it) in which every interval is finite. Suppose $A$ with partial order $\sqsubseteq$ is a causal set and that for all (not necessarily) distinct intervals $[p_1, q_1]$ and $[p_2, q_2]$ of $(R, A, A)$ that contain $x \in A$ there is an interval containing $x$ that is a subset of $[p_1, q_1] \cap [p_2, q_2]$. Show that the intersection of all intervals containing $x$ is an interval containing $x$. 
