

CIS/CSE 607 OPTIONAL Exam 1 Grade Booster  
May 3, 2009

General Instructions:

- Do all parts of all questions. If you find yourself writing long answers, you are probably not taking a good approach.
- DUE: Soft deadline: Saturday, May 9th, 11:59PM. You may submit by email or leave a written copy under my office door. If you submit by email, send a single .pdf file as an attachment. DO NOT send a .docx file, and DO NOT send a folder or .zip file to be processed. Do not send separate pages. Just one complete .pdf file or .doc file, not zipped. I'm leaving for a quantum computing conference on Sunday morning, May 10th. If you leave a paper copy under my office door, it must be there early on Sunday morning, May 10th. [Tip: If you create a .pdf file from a scan and it's too big to email, use a screen grabber to make .jpg's or .png's of each page. Usually the result is very much smaller - by as much as a factor of 10.]
- HARD deadline: I must submit grades by 11:59pm on Tuesday May 12th. I will check email just after 6pm, Monday, May 11th. If your problem set is not there, it will not be a factor in your course grade. So, please: the absolute email deadline is 6pm, Monday, May 11th.
- I will revise the grade of the first exam by averaging this optional problem set with the first exam, but I will give more weight to this problem set according to how much improvement is shown.
- You may email me with questions about the questions on this exam. I will answer in one of three general ways: (1) I may reply to you alone with a direct answer or hint. (2) I may reply to the whole class so that everyone benefits. (3) I will reply to you alone with a direct answer or hint in exchange for a small lowering of the grade on that problem. I will give you the choice first to accept a larger hint in exchange for a grade discount before proceeding, so that you have nothing to lose by sending a question by email.

**Problem 1.**

Let  $I$  be the identity function on a set  $A$ . That is,

$$I = \{(x, x) \mid x \in A\}$$

For each binary relation  $R$  on  $A$ , the *converse* of  $R$ , denoted by  $R^\smile$ , is given by

$$R^\smile = \{(y, x) \mid (x, y) \in R\}$$

Recall the definition of *composition* of binary relations on a set  $A$ : Let  $R$  and  $S$  be binary relations on set  $A$ . The *composition* of  $R$  followed by  $S$  is given by

$$S \circ R = \{(x, z) \mid \exists y [(x, y) \in R \wedge (y, z) \in S]\}$$

A binary relation  $R$  is *irreflexive* iff  $\forall x \in A [(x, x) \notin R]$ .

**Part 1:** Give an example of a set  $A$  and an irreflexive binary relation  $R$  on  $A$  such that  $R^\smile \circ R = I$ .

**Part 2:** A binary relation  $R$  on a set  $A$  is *symmetric* iff

$$\forall x \in A \forall y \in A [(x, y) \in R \supset (y, x) \in R].$$

Show that  $R$  is symmetric iff  $R = R^\smile$ .

**Part 3:** Give an example of a set  $A$  and a binary relation  $R$  on  $A$  such that  $R$  is not a function, and

$$R^\smile \circ R = I$$

**Part 4:** Suppose  $R$  is a binary relation on  $A$  and

$$R^\smile \circ R = R \circ R^\smile = I$$

Prove that  $R$  is a one-to-one and onto function.

**Problem 2.**

By definition, a relation binary relation  $R$  on a set  $A$  is *Euclidean* if, and only if,

$$\forall x \in A \forall y \in A \forall z \in A [\text{if } (x, y) \in R \wedge (x, z) \in R \text{ then } (y, z) \in R]$$

Give an argument that shows that every reflexive, Euclidean binary relation on a set  $A$  is also symmetric.

**Problem 3.**

Consider the following relation on the set of integers:

$$x \sim y \text{ iff } 9 \text{ divides } (x - y)$$

**Part 1:** Show that  $\sim$  is an equivalence relation.

**Part 2:** Go to

<http://www.learnenglish.org.uk/games/magic-gopher-central.swf>

Let  $n$  be your original number, and let  $s$  be the sum of its two digits. Show that

$$n \sim s$$

Look at the symbol in the gopher's chart next to each multiple of 9 from 9 to 81. Explain how the gopher always gets the correct symbol.

**Problem 4.** How many distinct digit-strings are there of length 9 that contain at least one occurrence of the digit 0 and at least one occurrence of the digit 9?

**Problem 5.** Consider the power set, i.e. the set of all subsets, of set  $S$ . Choose a particular subset  $P$  of  $S$ . For any two subsets  $A$  and  $B$  of  $S$ , let

$$A \sim_P B \text{ if, and only if, } A \cap P = B \cap P$$

**Part 1:** Show that  $\sim_P$  is an equivalence relation on the power set of  $S$ .

**Part 2:** Let  $S = \{1, 2, 3, 4\}$ , and let  $P = \{1, 3\}$ . List the equivalence classes determined by  $\sim_P$ . Note: For any subset  $A$  of  $S$ , the equivalence class of  $A$  is denoted by  $[A]$ , and

$$[A] = \{B \subseteq S \mid A \sim_P B\}.$$

**Part 3:** Again, let  $S$  be any set. Does the following rule (called a congruence) hold in general:

$$\text{if } A \sim_P A' \text{ and } B \sim_P B', \text{ then } (A \cup B) \sim_P (A' \cup B')$$