1. **Definition 0.1**: An *inner product* on a vector space V over the set of real numbers $\mathbb{R}$ is a function $\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$ that satisfies: for all vectors x, y and z in V and real numbers a and b in $\mathbb{R}$:

   (a) $\langle x, y \rangle = \langle y, x \rangle$

   (b) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

   (c) $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle$

   (d) $\langle x, x \rangle \geq 0$

   (e) $\langle x, x \rangle = 0$ iff $x = 0$.

   Note that in the last condition in the preceding definition $\langle x, x \rangle = 0$ iff $x = 0$, the occurrence of 0 on the left side of the equation refers to the real number 0, and the occurrence of 0 on the right side refers to the zero vector in V.

2. **Definition 0.2**: A *norm* on vector space V over $\mathbb{R}$ is a function $\| \cdot \| : V \longrightarrow \mathbb{R}$ that satisfies: for all x in V and a in $\mathbb{R}$:

   (a) $\|ax\| = |a|\|x\|$

   (b) $\|x + y\| \leq \|x\| + \|y\|$

   (c) $\|x\| = 0$ iff $x = 0$

   The first condition in the definition is called *first-order positive homogeneity*. The second condition is called *the triangle inequality* and, synonymously, *subadditivity*. The third condition is called *definiteness*.

3. **Task 1:** Show that first-order positive homogeneity and subadditivity together imply *positivity*: for all x in V, $\|x\| \geq 0$.

4. **Task 2:** Show that a norm can be defined from an inner product by letting

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}$$

5. **Definition 0.3**: A *metric* d on vector space V over $\mathbb{R}$ is a commutative, positive definite subadditive function $d : V \times V \longrightarrow \mathbb{R}$. i.e. d satisfies: for all x, y and z in V:

   (a) $d(x, y) = d(y, x)$ (commutativity)

   (b) $d(x, y) \geq 0$ (positivity)

(c) $d(x, y) = 0$ iff $x = y$ (definiteness)

(d) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

6. **Task 3:** Let $d : V \times V \longrightarrow \mathbb{R}$ be defined by

$$d(x, y) = \|x - y\|$$

   Show that d is a metric.

7. **Task 4:** Modify the IFS.java code posted on the course website to plot the result of the escape-time algorithm for the Mandelbrot set. Use 256 iterations as the cap on the number of iterations allowed for estimating membership in the Mandelbrot set.

8. **Task 5:** Modify the Rossler.java code posted on the course website to plot the return map f (also called a Lorenz map). There are multiple ways to define a return map. Here is one: let x_{old} be the value of x at some time t such that $\frac{dz}{dt} = 0$. Let x_{new} be the value of x the next time $\frac{dz}{dt} = 0$. Let $f(x_{\mathsf{old}}) = x_{\mathsf{new}}$. Replot f for the values of cR (see the code) 5.7, 6.2, 6.7, 7.2, 7.7, 8.0.

9. **Task 6:** Adapt the code you produced in Task 5 to plot the Feigenbaum diagram corresponding to f for all values of cR from 5.7 to 10.0