Discrete Mathematics Assessment Examination

Question 1: Let $\mathbf{N}$ be the set of nonnegative integers. Let $S = \{(0,0),(1,1)\}$. Prove: for every subset A of $\mathbf{N}$ and every subset B of $\mathbf{N}$, $S \neq A \times B$.

Hint: Assume $S = A \times B$ and try to get a contradiction.

Question 2: $A = \{0,1,2,3,4\}$. Let $B = \{0,1,2\}$. Let $g : B \longrightarrow A$ be specified by $g(x) = x$. How many functions $f : A \longrightarrow B$ are there such that the following condition is satisfied: $f(g(y)) = y$ for every y in B? Explain your reasoning. (Listing out every possibility **is** acceptable.)

Question 3: The Fibonacci numbers are generated by the rule

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-2} + F_{n-1}, \quad \text{for } n \geq 2.$$

The Lucas numbers are generated by the rule

$$L_0 = 2$$
$$L_1 = 1$$
$$L_n = L_{n-2} + L_{n-1}, \quad \text{for } n \geq 2.$$

Prove carefully by induction that for all $n \geq 1$, $L_n = F_{n-1} + F_{n+1}$.

Question 4: Let $f : X \longrightarrow Y$ and let $A \subseteq X$. Prove that

$$A \subseteq \hat{f}^{-1}(\hat{f}(A)).$$

For this question, refer to the appendix for an explanation of the notation. Pay attention to that little symbol ˆ that appears over some of the occurrences the f symbol here.

Question 5: Give an example of a function $f : X \longrightarrow Y$ and a subset A of X such that

$$A \neq \hat{f}^{-1}(\hat{f}(A)).$$

Again, refer to the appendix for an explanation of the notation.

Question 6:
a Explain why the function $h : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $h(x) = x^3 - x$ is surjective but not injective.
b Is the function $g : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $g(x) = 2^x$ injective, surjective, both (i.e. bijective), or neither?

Appendix

Definition 0.1 *For any sets A and B, the set $A \times B$, called the* Cartesion Product *of A and B, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words,*

$$A \times B = \{(a, b) \mid a \in A \textit{ and } b \in B\}.$$

Example 0.1 *Let $A = \{0, 1, 2\}$ and let $B = \{0, 1\}$. Then*

$$A \times B = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}.$$

Example 0.2 *Let $\mathbf{N}$ be the set of non-negative integers. That is, $\mathbf{N} = \{0, 1, 2, 3, \dots\}$. Then*

$$\mathbf{N} \times \mathbf{N} = \{(m, n) \mid m \in \mathbf{N} \textbf{ and } n \in \mathbf{N}\}.$$

Definition 0.2 *For any sets A and S, A is a* subset *of S if, and only if, every member of A is a member of S. We denote that A is a subset of S by $A \subseteq S$.*

Definition 0.3 *For any set S, the* power set *of S, denoted by $\mathbf{P}(S)$, is the set of all subsets of S. That is,*

$$\mathbf{P}(S) = \{B \mid B \subseteq S\}.$$

Note: The definition of power set implies that for anything x, $x \in \mathbf{P}(S)$ if, and only if, $x \subseteq S$.

Definition 0.4 *A* relation *R from set A to set B is a subset of $A \times B$.*

Definition 0.5 *A relation f from A to B is called a* function *from A to B if, and only if, for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$. Suppose f is a function from A to B. Given $a \in A$, the $b \in B$ for which $(a, b) \in f$ is denoted by $f(a)$. The expression $f : A \longrightarrow B$ means that f is a function from A to B.*

Definition 0.6 *A function $f : A \longrightarrow B$ is called an* injection*, if and only if the following condition for f is true: for all elements x and y of A such that $x \neq y$: $f(x) \neq f(y)$. When a function is an injection, we also say that the function is* injective*, and we also say that the function is* one-to-one*. These phrases are equivalent ways of expressing the same thing.*

Note: The definition says that $f : A \longrightarrow B$ is injective if, and only if, for every two different inputs to f we must get two different outputs. Equivalently, we cannot get the same output from f from two different inputs. The condition for f to be injective can be restated in the following equivalent form: for all elements x and y of A, if $f(x) = f(y)$, then $x = y$.

Definition 0.7 *A function $f : A \longrightarrow B$ is called a* surjection*, if and only if, the following condition for f is true: for each element b of B, there exists at least one $a \in A$ such that $f(a) = b$.*

Example 0.3 *Let $\mathbf{R}$ be the set of real numbers. The function $g : \mathbf{R} \longrightarrow \mathbf{R}$ that is specified by $g(x) = x^2$ is not surjective and is not injective. The function $h : \mathbf{R} \longrightarrow \mathbf{R}$ that is specified by $h(x) = x^3 - x$ is surjective but not injective. The function $exp : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $exp(x) = e^x$ is injective, but not surjective, and the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ specified by $f(x) = x^3$ is both injective and surjective.*

Definition 0.8 *Let $f : X \longrightarrow Y$. There are two functions associated with f that we will now define. The function $\hat{f} : \mathbf{P}(X) \longrightarrow \mathbf{P}(Y)$ is specified by*

$$\hat{f}(A) = \{y \in Y \mid \textit{for some } a \in A, y = f(a)\}.$$

The function $\hat{f}^{-1} : \mathbf{P}(Y) \longrightarrow \mathbf{P}(X)$ is specified by

$$\hat{f}^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Note: Do not jump to conclusions. The function $\hat{f}^{-1}$ is not the *inverse* of f or even the inverse of $\hat{f}$.